Viscoelasticity

Constitutive Modeling and FEM Implementation

MEC ENG 287: Introduction to Continuum Mechanics

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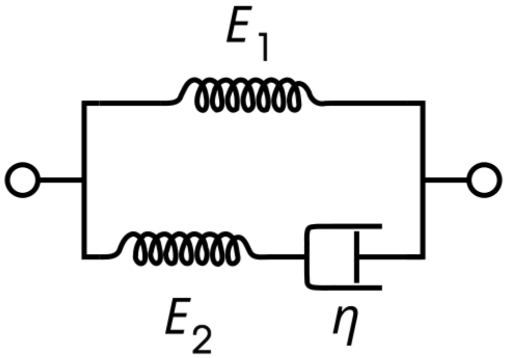
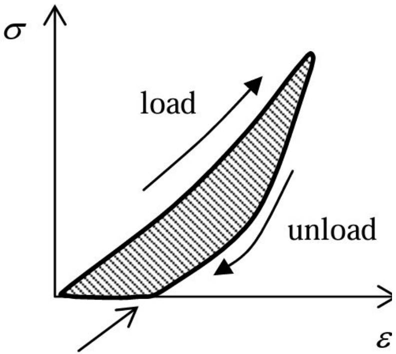
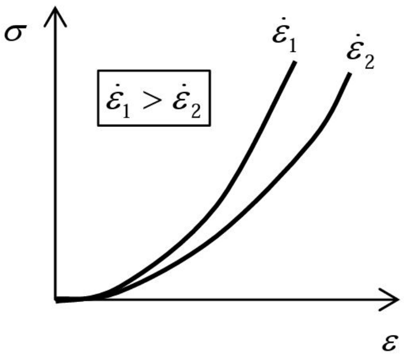
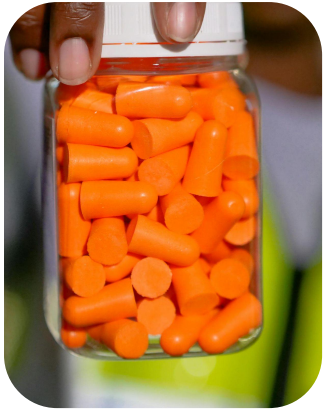
Background

- Viscoelasticity is the study of materials with a **time-dependence**.[2]
- “Visco” + “Elastic” solids return completely to their reference configuration when unloaded, but experience a time delay upon force unloading.
- Common viscoelastically modeled solids include: **polymers, biological tissues, and wood**.
- The standard 1D model of viscoelasticity is the Standard Linear Solid (SLS) model.







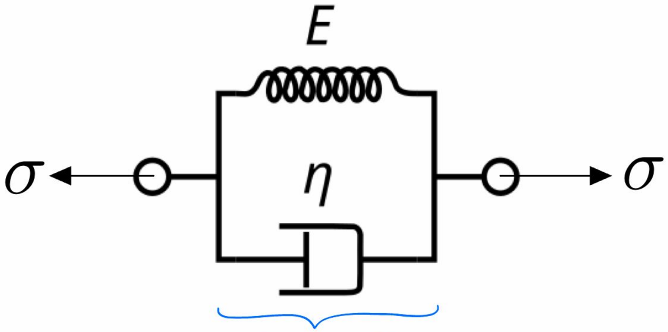


Earplugs (Polyurethane)

Wood (Cellulose fibers in polymeric lignin matrix)

Constitutive Analysis

Kelvin-Voigt 1D Analog



$$\varepsilon = \varepsilon^e + \varepsilon^v$$
$$\sigma = \sigma^e + \sigma^v = E\varepsilon^e + \eta\dot{\varepsilon}^v$$

- The stress is a (parallel) linear combination of elastic and viscous components
- As dashpots <u>cannot</u> store energy, the specific free energy is a function of only deformation and **not** the deformation rate

Initial constitutive dependences

Using the finite deformation isothermal hyperelastic model and Kelvin-Voigt analog model as guiding assumptions, we let both the specific free energy and Cauchy stress depend on both the deformation gradient **F** and the velocity gradient **L**.

$$\psi = \hat{\psi}(\mathbf{F}, \mathbf{L}), \ \mathbf{T} = \hat{\mathbf{T}}(\mathbf{F}, \mathbf{L})$$

Constitutive Analysis

Frame Indifference (1)

Constitutive functions should agree regardless of the relative motion between observers **O** and **O***, related by the defined motion vector.

$$\mathbf{r}^*(t) = \mathbf{y}(t) + \mathbf{Q}(t)\mathbf{r}(t)$$

$$\underline{\chi}^*(\mathbf{X}^*, t) = \mathbf{y}(t) + \mathbf{Q}(t)\underline{\chi}(\mathbf{X}, t)$$

$\mathbf{X}^*$ and $\mathbf{X}$ refer to the same material point p

$$\frac{\partial \underline{\chi}^*}{\partial \mathbf{X}} = \mathbf{F}^* = \frac{\partial \mathbf{r}^*}{\partial \mathbf{r}} \frac{\partial \mathbf{r}}{\partial \mathbf{X}} = \mathbf{Q}(t)\mathbf{F}$$

By taking the time derivative and utilizing the chain rule, we can also obtain the velocity gradient in frame **O***, **L***, in terms of **L.**

$$\underbrace{\dot{\mathbf{Q}}\mathbf{Q}^\top}_{=\underline{\Omega}} + \mathbf{Q}\mathbf{L}\mathbf{Q}^\top = \mathbf{L}^*$$

where $\underline{\Omega}$, a skew tensor representing "frame spin"

By frame indifference, we require that scalar and tensor state fields obey the frame relation:

$$\hat{\psi}^* = \hat{\psi}$$

$$\hat{\mathbf{T}}^* = \mathbf{Q}\hat{\mathbf{T}}\mathbf{Q}^\top$$

Constitutive Analysis

Frame Indifference (2)

With our proposed constitutive dependences:

$$\hat{\psi}(\mathbf{F}^*, \mathbf{L}^*) = \hat{\psi}(\mathbf{QF}, \underline{\Omega} + \mathbf{QLQ}^\top) = \hat{\psi}(\mathbf{F}, \mathbf{L})$$

$$\hat{\mathbf{T}}(\mathbf{F}^*, \mathbf{L}^*) = \hat{T}(\mathbf{QF}, \underline{\Omega} + \mathbf{QLQ}^\top) = \mathbf{Q}\hat{\mathbf{T}}(\mathbf{F}, \mathbf{L})\mathbf{Q}^\top$$

As with the isothermal compressible fluid derivation, the arbitrariness of **Ω** means that the skew part of **L** cannot affect either state field, reducing to a dependence on **D**.

As with hyperelasticity, by choosing $\mathbf{Q}=\mathbf{R}^\mathbf{T}$, we can show that dependence on only the symmetric stretch matrix component of **F** is equivalent to dependence on only $\mathbf{C}=\mathbf{F}^\mathbf{T}\mathbf{F}$.

We reduce our constitutive dependences:

$$\psi = \tilde{\psi}(\mathbf{C}, \mathbf{D}),\ \mathbf{T} = \tilde{\mathbf{T}}(\mathbf{C}, \mathbf{D})$$

To align our model notation with the Kelvin-Voigt analog, we can rewrite **D** using the identity:

$$\mathbf{D} = \frac{1}{2}\mathbf{F}^{-\top}\dot{\mathbf{C}}\mathbf{F}^{-1}$$

This allows us to express the state fields as functions of the deformation and deformation rate.

$$\psi = \tilde{\psi}(\mathbf{C}, \dot{\mathbf{C}}),\ \mathbf{T} = \tilde{\mathbf{T}}(\mathbf{C}, \dot{\mathbf{C}})$$

Constitutive Analysis

Thermodynamic Compatibility (1)

We start with the Second Law of Thermodynamics, with the isothermal assumption.

$$\rho\dot{\psi} - \mathbf{T} : \mathbf{D} \leq 0$$

We first want to find the material time derivative of the free energy, or $\boldsymbol{\psi}$. Through Frame Indifference, We concluded that the free energy is a function of the right Cauchy Green Deformation Tensor, **C**, and its material time derivative, **Ċ**.

$$\dot{\psi} = \frac{\partial \psi}{\partial \mathbf{C}} : \dot{\mathbf{C}} + \frac{\partial \psi}{\partial \dot{\mathbf{C}}} : \ddot{\mathbf{C}}$$

Now looking at the second term of the second law equation, we start by substituting **D** using the identity we introduced in Frame Indifference.

$$\mathbf{T} : \mathbf{D} = \mathbf{T} : \frac{1}{2}\mathbf{F}^{-T}\dot{\mathbf{C}}\mathbf{F}^{-1}$$

We then can commute the deformation gradient terms from the right side of the double dot over to the Cauchy Stress Tensor, resulting in the equation below

$$\frac{1}{2}\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} : \dot{\mathbf{C}}$$

Constitutive Analysis

Thermodynamic Compatibility (2)

By introducing the Second Piola Stress, **T$_{\mathbf{RR}}$**, we can rewrite the operations into the reference configuration.

$$\mathbf{T_{rr}} \equiv \mathbf{F}^{-1}\mathbf{T_r} = J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-\mathbf{T}}$$

$$\mathbf{T} : \mathbf{D} = \frac{1}{2}J^{-1}\mathbf{T_{rr}} : \dot{\mathbf{C}}$$

Through analyzing the Kelvin-Voigt 1D Analog, we expect an elastic term **independent** from a viscous term for our total stress. As a result, we will split the Second Piola Stress into two terms. The elastic term is the same as what we did in hyperelasticity and is solely dependent on **C**. The viscous term depends directly on **Ċ** and retains dependence on **C** for now.

$$\hat{\mathbf{T}}_{\mathbf{rr}}(\mathbf{C}, \dot{\mathbf{C}}) = \hat{\mathbf{T}}^{\mathbf{e}}_{\mathbf{rr}}(\mathbf{C}) + \hat{\mathbf{T}}^{\mathbf{v}}_{\mathbf{rr}}(\mathbf{C}, \dot{\mathbf{C}})$$

This expands the **T**:**D** operation to the following:

$$\frac{1}{2}J^{-1}(\hat{\mathbf{T}}^{\mathbf{e}}_{\mathbf{rr}}(\mathbf{C}) : \dot{\mathbf{C}} + \hat{\mathbf{T}}^{\mathbf{v}}_{\mathbf{rr}}(\mathbf{C}, \dot{\mathbf{C}}) : \dot{\mathbf{C}})$$

Constitutive Analysis

Thermodynamic Compatibility (3)

Combining the second law terms, we result in the following inequality:

$$(\rho \frac{\delta \hat{\psi}(\mathbf{C}, \dot{\mathbf{C}})}{\delta \mathbf{C}} - \frac{1}{2} J^{-1} \hat{\mathbf{T}}^{\mathbf{e}}_{\mathbf{rr}}(\mathbf{C}) : \dot{\mathbf{C}}) - \frac{1}{2} J^{-1} \hat{\mathbf{T}}^{\mathbf{v}}_{\mathbf{rr}}(\mathbf{C}, \dot{\mathbf{C}}) : \dot{\mathbf{C}} + \frac{\delta \hat{\psi}(\mathbf{C}, \dot{\mathbf{C}})}{\delta \dot{\mathbf{C}}} \ddot{\mathbf{C}} \leq 0$$

Now by utilizing the Coleman-Noll Procedure, we can derive the following identities:

$$\frac{\partial \hat{\psi}(\mathbf{C}, \dot{\mathbf{C}})}{\partial \dot{\mathbf{C}}} = 0$$

$$\hat{\mathbf{T}}^{\mathbf{e}}_{\mathbf{rr}} = 2 J \rho \frac{\delta \hat{\psi}(\mathbf{C})}{\delta \mathbf{C}} = 2 \frac{\delta \hat{\psi}_R(\mathbf{C})}{\delta \mathbf{C}}$$

$$\psi = \hat{\psi}(\mathbf{C})$$

$$\frac{1}{2} J^{-1} \hat{\mathbf{T}}^{\mathbf{v}}_{\mathbf{rr}} : \dot{\mathbf{C}} \geq 0$$

Constitutive Analysis

Thermodynamic Compatibility (4)

So far, we defined the Second Piola Stresses. Now, we want to compute the Cauchy Stresses in terms of more tangible components. Let's start with the elastic term.

$$\hat{\mathbf{T}}^{\mathbf{e}}_{\mathbf{rr}}(\mathbf{C}) = 2\frac{\delta\psi_R}{\delta\mathbf{C}} \quad , \quad J\mathbf{F}^{-1}\mathbf{T}^{\mathbf{e}}\mathbf{F}^{-\mathbf{T}} = 2\frac{\delta\psi_R}{\delta\mathbf{C}}$$

By inputting the Second Piola Stress Definition and commuting terms, we result in the elastic portion Cauchy Stress as follows:

$$\hat{\mathbf{T}}^{\mathbf{e}}(\mathbf{C}) = 2J^{-1}\mathbf{F}\frac{\delta\psi_R}{\delta\mathbf{C}}\mathbf{F}^{\mathbf{T}}$$

For the viscous portion, the inequality condition starts in terms of **Ċ**. We can rearrange the terms using the **Ċ** to **D** relation to show the viscous dissipation condition in the deformed state:

$$\mathbf{T}^{\mathbf{v}} : \mathbf{D} \geq 0$$

We define a function for $\mathbf{T}_{\mathbf{RR}}{}^{\mathbf{v}}$ that is linear with respect to **Ċ**. **T** is symmetric by balance of angular momentum and **Ċ** is symmetric, thus:

$$\mathbf{T}^{\mathbf{v}}_{\mathbf{rr}} = \lambda_1(tr\dot{\mathbf{C}})\mathbf{1} + \lambda_2\dot{\mathbf{C}}$$

Where the two λ are dependent on **C**. We can reduce dependence of the viscous stress solely to **Ċ** by assuming these λ are constants.

Constitutive Analysis

Thermodynamic Compatibility (5)

Substituting the linearized equation into the inequality we get the following:

$$\frac{1}{2}J^{-1}(\lambda_1(tr\dot{\mathbf{C}})\mathbf{1}+\lambda_2\dot{\mathbf{C}}):\dot{\mathbf{C}}\geq 0$$

Now, we can replace the $\dot{\mathbf{C}}$ terms with $\mathbf{D}$.

$$\dot{\mathbf{C}}=\mathbf{2F^TDF}$$

$$\frac{1}{2}J^{-1}(\lambda_1(tr\mathbf{2F^TDF})\mathbf{1}+\lambda_2\mathbf{2F^TDF}):\mathbf{2F^TDF}\geq 0$$

Through simplifying and commuting the F terms on the right, we get:

$$J^{-1}\mathbf{F}(\lambda_1(tr\mathbf{2F^TDF})\mathbf{1}+2\lambda_2\mathbf{F^TDF})\mathbf{F^T}:\mathbf{D}\geq 0$$

Now let’s first look at the λ_1 term. Through moving some terms around, we can get to this:

$$J^{-1}\mathbf{F}(\lambda_1(tr\mathbf{2F^TDF})\mathbf{1})\mathbf{F^T}=2J^{-1}\lambda_1(tr\mathbf{FF^TD})\mathbf{F1F^T}$$

Note that $\mathbf{B}=\mathbf{FF^T}$. Using this and the trace definition of contraction, we ultimately get this for the first term:

$$2J^{-1}\lambda_1(\mathbf{B}:\mathbf{D})\mathbf{B}$$

Constitutive Analysis

Thermodynamic Compatibility (6)

Now, let's move onto the λ_2 term. We can use the same logic as before

$$2J^{-1}\mathbf{F}(\lambda_2\mathbf{F^T D F})\mathbf{F^T} = 2J^{-1}\lambda_2\mathbf{F F^T D F F^T}$$

$$2J^{-1}\lambda_2\mathbf{BDB}$$

Now, we can combine the terms in order to get the viscous portion of the inequality in terms of Cauchy Stress.

$$2J^{-1}(\lambda_1(\mathbf{B}:\mathbf{D})\mathbf{B} + \lambda_2\mathbf{BDB}):\mathbf{D} \geq 0$$

However, we can simplify even further! By looking at the following statement, we can also determine that both λ_1 and λ_2 must both be greater or equal than zero in order to satisfy all values of **B** and **D**.

$$2J^{-1}(\lambda_1(\mathbf{B}:\mathbf{D})^2 + \lambda_2 tr(\mathbf{BDBD})) \geq 0$$

The inequality we created also shows us the viscous portion of the Cauchy Stress. Unlike the elastic portion, the viscous portion **does not** follow the direction of **D**!

$$\hat{\mathbf{T}}_{\mathbf{vis}}(\mathbf{C},\dot{\mathbf{C}}) = 2J^{-1}(\lambda_1(\mathbf{B}:\mathbf{D})\mathbf{B} + \lambda_2\mathbf{BDB})$$

For a small strain assumption, J ≈ 1 and **B** ≈ **1**. The viscous Cauchy Stress is then shown to be as follows:

$$\mathbf{T}_{\mathrm{vis}} \approx 2(\lambda_1(\mathrm{tr}\,\mathbf{D})\mathbf{1} + \lambda_2\mathbf{D})$$

Constitutive Analysis

Thermodynamic Compatibility (7)

Now that we computed both the viscous and elastic portion of the Cauchy Stress, we can combine the terms!

$$\hat{\mathbf{T}}(\mathbf{C},\dot{\mathbf{C}}) = \hat{\mathbf{T}}^{\mathbf{e}}(\mathbf{C}) + \hat{\mathbf{T}}^{\mathbf{v}}(\mathbf{C},\dot{\mathbf{C}}) = 2J^{-1}(\mathbf{F}\frac{\delta\psi_R}{\delta\mathbf{C}}\mathbf{F}^{\mathbf{T}} + \lambda_1(\mathbf{B}:\mathbf{D})\mathbf{B} + \lambda_2\mathbf{BDB})$$

How would the viscous portion affect other directions?

Ex. Necking in silly putty often occurs whenever you slowly pull the putty apart. Whenever necking occurs, the putty gets thinner at a much faster pace than from elastic deformation. This is a result of the Viscous Cauchy Stress affecting other directions!

Constitutive Analysis

Material Symmetry Considerations (1)

With our derived constraints on **T^e^** and **T^v^** that parallel hyperelastic and viscous fluid models, we can construct similar isotropic relations. For example, assume the material has an isotropic **specific energy** such that ψ**(C)** is the same for **C’** as defined by **F’** = **FQ** for any choice of **Q**:

$$
\begin{aligned}
\hat{\psi}_R(\mathbf{C}) &= \hat{\psi}_R(\mathbf{F}^{\mathrm{T}}\mathbf{F}) \\
= \hat{\psi}_R(\mathbf{C}') &= \hat{\psi}_R(\mathbf{Q}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{Q})
\end{aligned}
$$

$$
\left[\hat{\psi}_R(\mathbf{C}) = \hat{\psi}_R(\mathbf{Q}^{\mathrm{T}}\mathbf{C}\mathbf{Q})\right]
$$

Next, derive the isotropic **elastic** constraint:

$$
\begin{aligned}
\mathbf{T}^{\mathrm{e}} &= \hat{\mathbf{T}}^{\mathrm{e}}(\mathbf{F}) = J^{-1}\mathbf{F}\hat{\mathbf{T}}^{\mathrm{e}}{}_{\mathbf{RR}}(\mathbf{C})\mathbf{F}^{\mathrm{T}} \\
&= \hat{\mathbf{T}}^{\mathrm{e}}(\mathbf{F}\mathbf{Q}) = J^{-1}\mathbf{F}\mathbf{Q}\hat{\mathbf{T}}^{\mathrm{e}}{}_{\mathbf{RR}}(\mathbf{Q}^{\mathrm{T}}\mathbf{C}\mathbf{Q})\mathbf{Q}^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}
\end{aligned}
$$

$$
\left[\mathbf{Q}^{\mathrm{T}}\hat{\mathbf{T}}^{\mathrm{e}}{}_{\mathbf{RR}}(\mathbf{C})\mathbf{Q} = \hat{\mathbf{T}}^{\mathrm{e}}{}_{\mathbf{RR}}(\mathbf{Q}^{\mathrm{T}}\mathbf{C}\mathbf{Q})\right]
$$

Constitutive Analysis

Material Symmetry Considerations (2)

For **T^v**, we maintain the same linear viscosity assumption as before such that:

$$\mathbf{T}^{\mathbf{v}}_{\mathbf{rr}} = \lambda_1(tr\dot{\mathbf{C}})\mathbf{1} + \lambda_2\dot{\mathbf{C}}$$

Or equivalently:

$$\hat{\mathbf{T}}^{\mathbf{v}}(\mathbf{D}) = \mathcal{C}\mathbf{D}$$

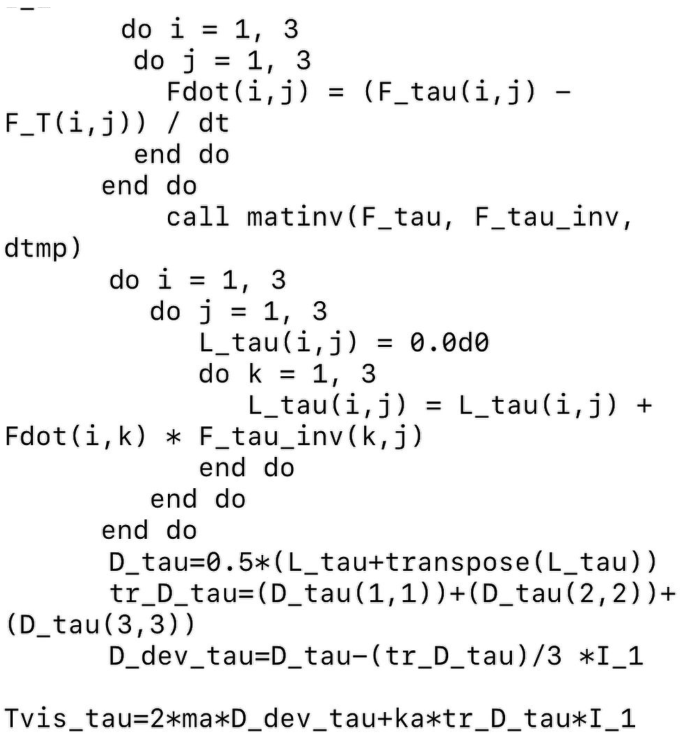
Where $\mathcal{C}$ is a fourth order tensor. As with an linearly viscous fluid, by utilizing symmetry of **D** and **T^v** we can derive the equivalent relations

$$\hat{\mathbf{T}}^{\mathbf{v}}(\mathbf{D}) = 2\mu\mathbf{D} + \lambda(tr\mathbf{D})\mathbf{1}$$

$$\hat{\mathbf{T}}^{\mathbf{v}}(\mathbf{D}) = 2\mu\mathbf{D}_0 + \kappa(tr\mathbf{D})\mathbf{1}$$

For any constant choice of **Q**, **D** and **Ċ** remain unchanged. That is, just as with linearly viscous fluids, the viscous stress **T^v** is **inherently** isotropic given our choice of constitutive relation.

VUMAT Modifications



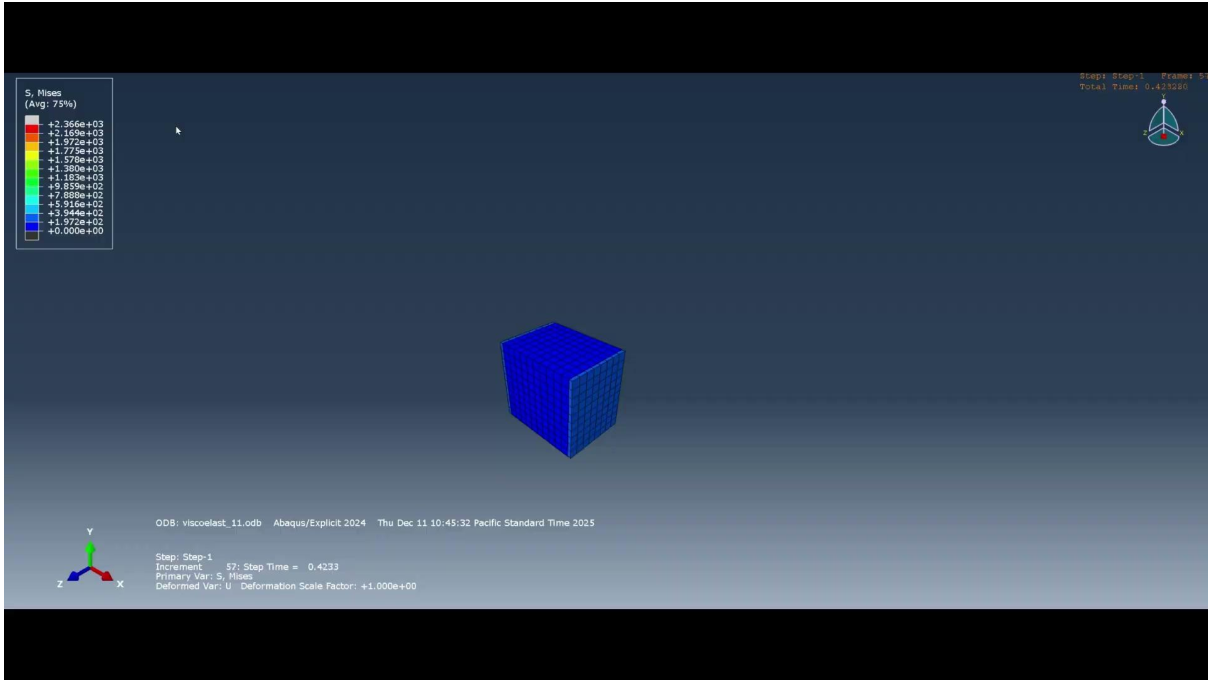
```
      do i = 1, 3
       do j = 1, 3
         Fdot(i,j) = (F_tau(i,j) -
F_T(i,j)) / dt
       end do
     end do
         call matinv(F_tau, F_tau_inv,
dtmp)
      do i = 1, 3
        do j = 1, 3
          L_tau(i,j) = 0.0d0
          do k = 1, 3
             L_tau(i,j) = L_tau(i,j) +
Fdot(i,k) * F_tau_inv(k,j)
          end do
        end do
     end do
      D_tau=0.5*(L_tau+transpose(L_tau))
      tr_D_tau=(D_tau(1,1))+(D_tau(2,2))+
(D_tau(3,3))
      D_dev_tau=D_tau-(tr_D_tau)/3 *I_1

Tvis_tau=2*ma*D_dev_tau+ka*tr_D_tau*I_1
```

```
   T_tau = matmul(Re_tau,
matmul(Te_0+I_1*Te_sph,
     +   transpose(Re_tau)))/det_Fe
+Tvis_tau
```

Numerical Tests

One-Element Implementation



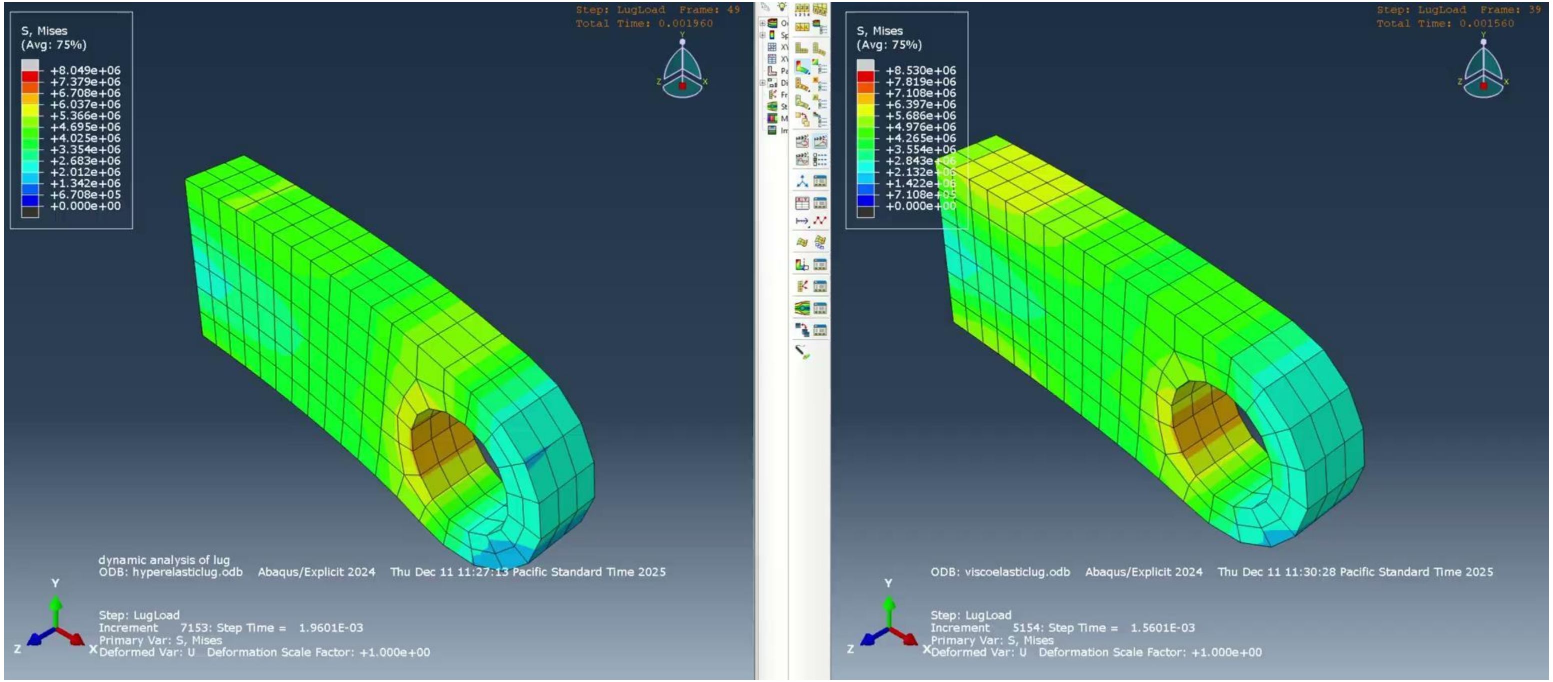
```
Step: Step-1   Frame: 5
Total Time: 0.423280
```

```
ODB: viscoelast_11.odb    Abaqus/Explicit 2024    Thu Dec 11 10:45:32 Pacific Standard Time 2025

Step: Step-1
Increment      57: Step Time =    0.4233
Primary Var: S, Mises
Deformed Var: U   Deformation Scale Factor: +1.000e+00
```

Numerical Tests

Hyperelastic Geometry Implementation



Step: LugLoad Frame: 49
Total Time: 0.001960

dynamic analysis of lug
ODB: hyperelasticlug.odb Abaqus/Explicit 2024 Thu Dec 11 11:27:13 Pacific Standard Time 2025

Step: LugLoad
Increment 7153: Step Time = 1.9601E-03
Primary Var: S, Mises
Deformed Var: U Deformation Scale Factor: +1.000e+00

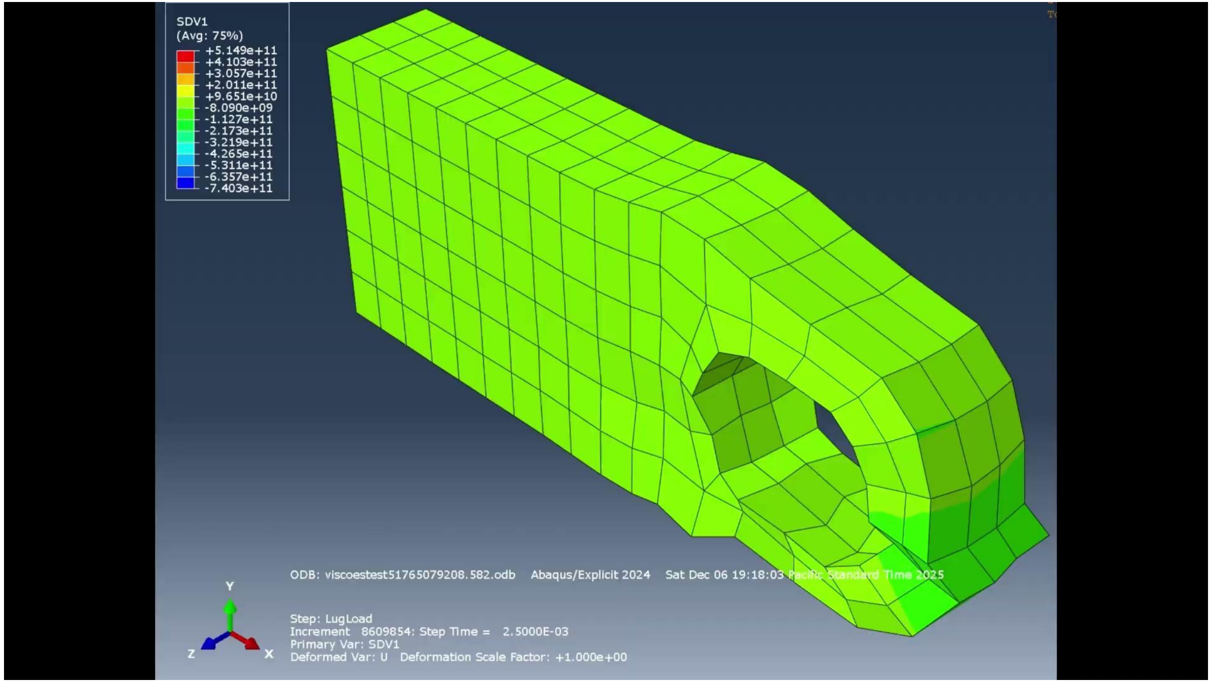
Step: LugLoad Frame: 39
Total Time: 0.001560

ODB: viscoelasticlug.odb Abaqus/Explicit 2024 Thu Dec 11 11:30:28 Pacific Standard Time 2025

Step: LugLoad
Increment 5154: Step Time = 1.5601E-03
Primary Var: S, Mises
Deformed Var: U Deformation Scale Factor: +1.000e+00

Numerical Tests

Viscoelastic Geometry Implementation



```
ODB: viscoestest51765079208.582.odb    Abaqus/Explicit 2024    Sat Dec 06 19:18:03 Pacific Standard Time 2025

Step: LugLoad
Increment   8609854: Step Time =   2.5000E-03
Primary Var: SDV1
Deformed Var: U   Deformation Scale Factor: +1.000e+00
```

References

[1] Anand, Lallit, Ken Kamrin, and Sanjay Govindjee, 'Linear viscoelasticity', *Introduction to Mechanics of Solid Materials* (Oxford, 2022; online edn, Oxford Academic, 19 Jan. 2023), https://doi.org/10.1093/oso/9780192866073.003.0019, accessed 9 Dec. 2025.

[2] Kelly, Piaras, *Solid Mechanics Part I: An Introduction to Solid Mechanics* (University of Auckland; online edn, Department of Engineering Science, https://pkel015.connect.amazon.auckland.ac.nz/SolidMechanicsBooks/Part_I/index.html, accessed 9 Dec. 2025.