Generalized Continuum Mechanics and the Virtual Powe

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1 Introduction

To properly contextualize the advanced theories discussed in this work, it is necessary to first establish the foundational axioms of classical continuum mechanics. The classical theory begins with Leonhard Euler's notion of contiguity, which treats a material body not as a discrete collection of atoms, but as a continuous mass distributed continuously throughout a region of space. From this premise, the global statements for the balance of linear momentum and the balance of angular momentum are postulated for any arbitrary subregion of the body.

The mathematical formalization of internal forces within this continuum was achieved by Augustin-Louis Cauchy, who generalized Euler's concept of hydrostatic fluid pressure into the Cauchy stress tensor. By applying Green's Divergence Theorem (also known as Gauss's theorem), the global integral laws of momentum balance are elegantly transformed into local, partial differential equations governing the equilibrium and motion of the continuum at every point.

However, this classical formulation is inherently restrictive. It relies on a strict set of underlying assumptions:

1. **Absence of Volume Couples:** It is assumed that there is no density of volume couples (distributed body moments) acting on the material.
2. **Symmetry of Stress:** As a direct mathematical consequence of the balance of angular momentum and the absence of volume couples, the Cauchy stress tensor is forced to be strictly symmetric.
3. **Force and Acceleration:** Standard body forces (such as gravity) are permitted, and accelerations are prescribed purely by the classical mechanics of Newton, Lagrange, and Cauchy.
4. **Spatiotemporal Axioms:** All physical processes occur within a standard three-dimensional Euclidean space and absolute time.
5. **Absence of Singularities:** The mathematical fields describing the continuum (such as displacement and stress) are assumed to be sufficiently smooth; discrete micro-defects, cracks, and singularities are effectively smeared out or ignored.

Any theoretical framework that intentionally relaxes or abandons one or more of these classical assumptions falls under the umbrella of **Generalized Continuum Mechanics**. These generalizations are typically required when the macroscopic behavior of a material is heavily influenced by its internal microstructure or by phenomena occurring at internal characteristic length scales.

In the following sections, we explore several prominent generalized methods, highlighting their specific physical assumptions and detailing exactly which classical laws they break.

1.1 Weak Nonlocality: Strain Gradient Theories

Classical continuum mechanics relies on the principle of local action, which dictates that the material state at a specific point depends solely on the thermomechanical history of that exact point. Weakly nonlocal theories break this assumption by proposing that the material state is also influenced by the state of its immediate neighborhood. Mathematically, this is achieved by introducing spatial gradients of strain into the constitutive equations.

The Aifantis Theory: The Aifantis model is a phenomenological approach to gradient plasticity. It introduces weak nonlocality by incorporating the Laplacian of the accumulated plastic strain into the yield condition. While it successfully predicts spatial patterning and size effects without abandoning the classical codirectionality hypothesis (where plastic flow aligns perfectly with deviatoric stress), its reliance on a scalar gradient limits its ability to model complex, directionally dependent internal stresses.

The Gurtin and Anand Theory: In contrast, the theory proposed by Gurtin and Anand is a thermodynamically rigorous, tensorial gradient framework based on the Principle of Virtual Power. It introduces independent microscopic hyperstresses that are power-conjugate to the gradient of the plastic strain rate. By defining the material's defect energy in terms of the Burgers tensor (a macroscopic measure of geometrically necessary dislocations), this theory elegantly captures complex directional backstresses and the resulting size-dependent strengthening, fundamentally departing from purely local plasticity.

1.2 Cosserat (Micropolar) Media

The Cosserat theory breaks the classical assumption that material points are simply structureless geometrical dots capable only of translation. Instead, a Cosserat continuum models material points as infinitesimal rigid bodies (or triads) that possess independent rotational degrees of freedom, termed *microrotation.*

Because the microstructure can spin independently of the macroscopic material deformation, the theory must introduce couple-stresses (moments per unit area) to resist these internal rotations. Consequently, the assumption of zero volume couples is broken, allowing for the existence of external body couples. Most importantly, the presence of couple-stresses fundamentally alters the balance of angular momentum, meaning **the Cauchy stress tensor is no longer required to be symmetric**. This asymmetry is the hallmark of Cosserat media, allowing it to efficiently model granular materials, foams, and bone, where particle rotations transfer significant load.

1.3 Micromorphic and Microstructure Media

Micromorphic theory, pioneered by Mindlin and Eringen, represents an even more radical departure from classical mechanics. While Cosserat theory allows the microstructure to rotate rigidly, micromorphic theory breaks the assumption of microstructural rigidity entirely.

In a micromorphic continuum, the material points are treated as deformable micro-volumes that can independently stretch, shear, and rotate, completely decoupled from the macroscopic deformation of the bulk material. To balance these highly complex internal kinematics, the theory introduces an extensive suite of generalized micro-stresses and higher-order stress measures. It breaks virtually all local assumptions of classical kinematics, making it the most mathematically complete (and computationally demanding) framework for modeling highly heterogeneous materials with complex internal substructures.

2 Conventional Plasticity via the Principle of Virtual Power

1. Kinematic Descriptors

In conventional plasticity, the basic "rate-like" descriptors are the velocity $\dot{u}$, the elastic distortion-rate H^e, and the plastic strain-rate H^p. These are constrained by the following kinematic relationship:

$$\nabla \dot{u} = H^e + H^p \tag{1}$$

Because plastic flow represents the volume-preserving motion of dislocations through the material structure, the plastic strain rate is deviatoric (trace-free):

$$\mathrm{tr}(H^p) = 0 \tag{2}$$

2. The Principle of Virtual Power (Integral Form)

The virtual power formulation replaces the classical macroscopic stress power with a detailed reckoning of two disparate processes: the elastic stretching of the microscopic structure and the plastic flow of dislocations.

Internal Power: We allow for power expended internally by an elastic stress T^e (conjugate to the elastic rate) and a plastic stress T^p (conjugate to the plastic rate).

$$\mathcal{I}(P) = \int_P \Big(T^e : \tilde{H}^e + T^p : \tilde{H}^p \Big)\, dv \tag{3}$$

External Power: Power is expended externally on the subregion P by surface tractions $t(n)$ acting over the boundary ∂P and body forces b acting within P.

$$\mathcal{W}(P) = \int_{\partial P} t(n) \cdot \tilde{u}\, da + \int_P b \cdot \tilde{u}\, dv \tag{4}$$

The Principle of Virtual Power requires that for any subregion P and any virtual velocity fields, the external power must balance the internal power ($\mathcal{W} = \mathcal{I}$):

$$\int_{\partial P} t(n) \cdot \tilde{u}\, da + \int_P b \cdot \tilde{u}\, dv = \int_P \Big(T^e : \tilde{H}^e + T^p : \tilde{H}^p \Big)\, dv \tag{5}$$

3. Going from Integral Form to Differential Form

To find the local equations governing the material, we apply the integral balance using specific, clever choices for our virtual velocities.

A. The Macroscopic Force Balance

We choose a "macroscopic" virtual velocity where $\tilde{H}^p = 0$, which forces $\tilde{H}^e = \nabla \tilde{u}$. Substituting this into the power balance gives:

$$\int_{\partial P} t(n) \cdot \tilde{u}\, da + \int_P b \cdot \tilde{u}\, dv = \int_P T^e : \nabla \tilde{u}\, dv \tag{6}$$

To extract the differential equation, we must use the **Divergence Theorem** on the right side of the equation. We use the identity $\mathrm{Div}(T^T \tilde{u}) = T : \nabla \tilde{u} + (\mathrm{Div}\, T) \cdot \tilde{u}$:

$$\int_P T^e : \nabla \tilde{u}\, dv = \int_{\partial P} (T^e n) \cdot \tilde{u}\, da - \int_P (\mathrm{Div}\, T^e) \cdot \tilde{u}\, dv \tag{7}$$

Substituting this back into our balance and rearranging terms yields:

$$\int_{\partial P} (t(n) - T^e n) \cdot \tilde{u}\, da + \int_P (\mathrm{Div}\, T^e + b) \cdot \tilde{u}\, dv = 0 \tag{8}$$

Because this must hold true for *any* arbitrary virtual velocity $\tilde{u}$ and *any* arbitrary subregion P, both integrands must independently equal zero (via the fundamental lemma of the calculus of variations). This gives us our classical macroscopic conditions (where $T = T^e$):

Traction Condition:

$$t(n) = Tn \tag{9}$$

Macroscopic Force Balance:

$$\mathrm{Div}\, T + b = 0 \tag{10}$$

B. The Microscopic Force Balance

Next, we choose a "microscopic" virtual velocity where there is no macroscopic motion ($\tilde{u} = 0$), forcing $\tilde{H}^e = -\tilde{H}^p$. The external power vanishes, and the balance reduces to:

$$\int_P (T^p - T) : \tilde{H}^p\, dv = 0 \tag{11}$$

Because T is symmetric and $\tilde{H}^p$ is deviatoric, only the deviatoric part of the macroscopic stress (T_0) does work. Since this must hold for any arbitrary plastic rate, we obtain the **Microscopic Force Balance**:

$$T_0 = T^p \tag{12}$$

4. Going from Differential Form to Integral Form

The integral power balance is fundamentally equivalent to the classical local balance $\operatorname{Div} T + b = 0$. Here are the exact mathematical steps to prove this equivalence in reverse.

1. **Start with the local balance equation:**

$$\operatorname{Div} T + b = 0 \tag{13}$$

2. **Multiply (dot product) by an arbitrary virtual velocity field $\tilde{u}$:**

$$(\operatorname{Div} T) \cdot \tilde{u} + b \cdot \tilde{u} = 0 \tag{14}$$

3. **Integrate the entire equation over a volume P:**

$$\int_P (\operatorname{Div} T) \cdot \tilde{u}\, dv + \int_P b \cdot \tilde{u}\, dv = 0 \tag{15}$$

4. **Apply the standard Divergence Theorem to the first boundary term:**

$$\int_{\partial P} (Tn) \cdot \tilde{u}\, da - \int_P T : \nabla \tilde{u}\, dv + \int_P b \cdot \tilde{u}\, dv = 0 \tag{16}$$

5. **Substitute the Cauchy stress theorem (boundary condition):** We know the traction vector on the surface is defined as $t(n) = Tn$.

$$\int_{\partial P} t(n) \cdot \tilde{u}\, da - \int_P T : \nabla \tilde{u}\, dv + \int_P b \cdot \tilde{u}\, dv = 0 \tag{17}$$

6. **Rearrange to recover the classical Principle of Virtual Power:**

$$\int_{\partial P} t(n) \cdot \tilde{u}\, da + \int_P b \cdot \tilde{u}\, dv = \int_P T : \nabla \tilde{u}\, dv \tag{18}$$

5. Adding the Codirectionality Constraint

A more restrictive, yet widely used, form of the virtual-power principle requires from the outset that the codirectionality constraint be satisfied. This constraint mandates that the direction of plastic flow coincides with the direction of the deviatoric stress:

$$\frac{T_0}{|T_0|} = N^p \quad \text{and} \quad N^p = \frac{\dot{E}^p}{|\dot{E}^p|} \tag{19}$$

By defining the accumulated plastic strain rate as $\dot{e}^p = |\dot{E}^p| \geq 0$, we can rewrite the plastic strain rate simply as $\dot{E}^p = \dot{e}^p N^p$.

Assuming an irrotational plastic flow (zero plastic spin), we can establish a streamlined virtual-power principle based on this codirectionality.

Streamlined Kinematics: The basic kinematic constraint reduces to utilize a scalar plastic flow multiplier:

$$\operatorname{sym} \nabla \dot{u} = \dot{E}^e + \dot{e}^p N^p \tag{20}$$

Streamlined Internal Power: We define the scalar resolved plastic stress as $\tau^p = N^p : T^p$. The internal power expenditure is then simplified to account for work done along this specific flow direction:

$$\mathcal{I}(P) = \int_P \left(T : \dot{E}^e + \tau^p \dot{e}^p \right) dv \tag{21}$$

Scalar Microscopic Force Balance: Applying the principle of virtual power to this streamlined form yields a new microscopic balance. By choosing a microscopic virtual velocity where $\tilde{u} = 0$ and $\tilde{E}^e = -\tilde{e}^p N^p$, the power balance reduces to an equation involving only the scalar magnitudes. Since $\tilde{e}^p \geq 0$ is arbitrary, we obtain the scalar microscopic force balance:

$$|T_0| = \tau^p \tag{22}$$

An interesting consequence of the codirectionality constraint is the scalar nature of this balance; only the normal part of T^p is mechanically active, rendering the tangential and skew parts indeterminate or zero.

3 The Gradient Theory of Aifantis

The earliest attempt at a plasticity theory with a material length-scale is contained in the work of Aifantis. Working within the framework of small deformations, Aifantis proposed a flow rule obtained by adding a Laplacian term to the conventional flow resistance. In this theory, the relevant gradient field is the gradient of the accumulated plastic strain, and a basic assumption is the conventional codirectionality hypothesis.

3.1 Kinematic Descriptors

In the Aifantis theory, the basic kinematical descriptors are the velocity $\dot{\mathbf{u}}$, the elastic strain-rate $\dot{\mathbf{E}}^e$, and the rate $\dot{e}^p$ of the accumulated plastic strain. These descriptors are subject to the kinematical constraint:

$$\operatorname{sym} \nabla\dot{\mathbf{u}} = \dot{\mathbf{E}}^e + \dot{e}^p \mathbf{N}^p, \tag{23}$$

where the flow direction $\mathbf{N}^p$ is consistent with the codirectionality constraint:

$$\frac{\mathbf{T}_0}{|\mathbf{T}_0|} = \mathbf{N}^p. \tag{24}$$

Necessarily, the conventional stress $\mathbf{T}$ is symmetric. We restrict our attention to irrotational plastic flow, where the plastic spin $\mathbf{W}^p \equiv \mathbf{0}$.

3.2 Internal Power

Because our goal is a theory that accounts explicitly for the gradient of the accumulated plastic strain, we allow for power expended internally by a microscopic hyperstress $\boldsymbol{\xi}^p$ power-conjugate to $\nabla\dot{e}^p$. The internal power is written in the form:

$$\mathcal{I}(P) = \int_P \Big(\mathbf{T}{:}\dot{\mathbf{E}}^e + \tau^p \dot{e}^p + \boldsymbol{\xi}^p \cdot \nabla\dot{e}^p\Big)\, dv. \tag{25}$$

3.3 External Power

This internal power must be balanced by power expended externally by tractions on the boundary ∂P and body forces acting within P. The conventional traction $\mathbf{t}(\mathbf{n})$ is not sufficiently general to accommodate the internal power. Guided by the divergence theorem, we assume that power is expended externally by a microscopic hypertraction $\chi(\mathbf{n})$ conjugate to $\dot{e}^p$. The external power has the form:

$$\mathcal{W}(P) = \int_{\partial P} \mathbf{t}(\mathbf{n}) \cdot \dot{\mathbf{u}}\, da + \int_P \mathbf{b} \cdot \dot{\mathbf{u}}\, dv + \int_{\partial P} \chi(\mathbf{n})\dot{e}^p\, da, \tag{26}$$

with $\chi(\mathbf{n})$ defined over the body for all time.

3.4 Power Balance and the Principle of Virtual Power

Our discussion of virtual power is therefore based on the power balance equating the external and internal power:

$$\underbrace{\int_{\partial P} (\mathbf{t}(\mathbf{n}) \cdot \dot{\mathbf{u}} + \chi(\mathbf{n})\dot{e}^p)\, da + \int_P \mathbf{b} \cdot \dot{\mathbf{u}}\, dv}_{\mathcal{W}(P)} = \underbrace{\int_P \Big(\mathbf{T}{:}\dot{\mathbf{E}}^e + \tau^p \dot{e}^p + \boldsymbol{\xi}^p \cdot \nabla\dot{e}^p\Big)\, dv}_{\mathcal{I}(P)}. \tag{27}$$

We use the term virtual velocity for a list $\mathcal{V} = (\tilde{\mathbf{u}}, \tilde{\mathbf{E}}^e, \tilde{e}^p)$ consistent with the kinematical constraint. The principle requires that for any subregion P and any choice of the virtual velocity $\mathcal{V}$:

$$\underbrace{\int_{\partial P} (\mathbf{t}(\mathbf{n}) \cdot \tilde{\mathbf{u}} + \chi(\mathbf{n})\tilde{e}^p)\, da + \int_P \mathbf{b} \cdot \tilde{\mathbf{u}}\, dv}_{\mathcal{W}(P,\mathcal{V})} = \underbrace{\int_P \Big(\mathbf{T}{:}\tilde{\mathbf{E}}^e + \tau^p \tilde{e}^p + \boldsymbol{\xi}^p \cdot \nabla \tilde{e}^p\Big)\, dv}_{\mathcal{I}(P,\mathcal{V})}\,. \tag{28}$$

3.5 Force Balances

3.5.1 Balance 1: Macroscopic Force Balance

Consider a virtual velocity $\mathcal{V}$ with $\tilde{\mathbf{E}}^e = \operatorname{sym} \nabla\tilde{\mathbf{u}}$, so that by the kinematical constraint $\tilde{e}^p \equiv 0$. Since $\mathbf{T}$ is symmetric, $\mathbf{T}{:}\tilde{\mathbf{E}}^e = \mathbf{T}{:}\nabla\tilde{\mathbf{u}}$, and the balance becomes:

$$\int_{\partial P} \mathbf{t}(\mathbf{n}) \cdot \tilde{\mathbf{u}}\, da + \int_P \mathbf{b} \cdot \tilde{\mathbf{u}}\, dv = \int_P \mathbf{T}{:}\nabla\tilde{\mathbf{u}}\, dv. \tag{29}$$

This balance is required to hold for all $\tilde{\mathbf{u}}$ and P. Its consequences are the macroscopic traction condition:

$$\mathbf{t}(\mathbf{n}) = \mathbf{T}\mathbf{n} \tag{30}$$

and the macroscopic force balance:

$$\operatorname{Div} \mathbf{T} + \mathbf{b} = \mathbf{0}. \tag{31}$$

3.5.2 Balance 2: Microscopic Force Balance

We define the resolved shear-stress through $\tau = \mathbf{T}_0{:}\mathbf{N}^p$. Consider a virtual velocity $\mathcal{V}$ with $\tilde{\mathbf{u}} \equiv \mathbf{0}$. Choose the virtual field $\tilde{e}^p$ arbitrarily and let $\tilde{\mathbf{E}}^e = -\tilde{e}^p\mathbf{N}^p$. Then, $\mathbf{T}{:}\tilde{\mathbf{E}}^e = -\tau\tilde{e}^p$. Using the divergence theorem, we find that:

$$\int_{\partial P} (\chi(\mathbf{n}) - \boldsymbol{\xi}^p \cdot \mathbf{n})\, \tilde{e}^p\, da + \int_P (\tau - \tau^p + \operatorname{Div} \boldsymbol{\xi}^p)\, \tilde{e}^p\, dv = 0. \tag{32}$$

A standard argument based on the fundamental lemma of the calculus of variations yields the microscopic traction condition:

$$\chi(\mathbf{n}) = \boldsymbol{\xi}^p \cdot \mathbf{n} \tag{33}$$

and the microscopic force balance:

$$\tau = \tau^p - \operatorname{Div} \boldsymbol{\xi}^p. \tag{34}$$

3.6 Free-Energy Imbalance

Arguing as in classical thermodynamics, the temporal increase in free energy of P must be balanced by the power expended on P minus the nonnegative dissipation δ. This yields the local free-energy imbalance:

$$\dot{\Psi} - \mathbf{T}{:}\dot{\mathbf{E}}^e - \tau^p \dot{e}^p - \boldsymbol{\xi}^p \cdot \nabla \dot{e}^p = -\delta \le 0. \tag{35}$$

3.7 Constitutive Equations

We assume that the free energy admits a decomposition $\Psi = \Psi^e + \Psi^p$, in which Ψ^e represents elastic strain energy, while $\Psi^p = \hat{\Psi}^p(e^p, \mathbf{g}^p)$ is a defect energy. Here, $\mathbf{g}^p = \nabla e^p$.

We introduce energetic microscopic stresses τ^p_{en} and $\boldsymbol{\xi}^p_{en}$ through the relations:

$$\tau^p_{en} = \frac{\partial \hat{\Psi}^p(e^p, \mathbf{g}^p)}{\partial e^p}, \qquad \boldsymbol{\xi}^p_{en} = \frac{\partial \hat{\Psi}^p(e^p, \mathbf{g}^p)}{\partial \mathbf{g}^p}. \tag{36}$$

Substituting these into the free-energy imbalance, we find that the dissipation δ is given by:

$$\delta = (\tau^p - \tau^p_{en})\dot{e}^p + (\boldsymbol{\xi}^p - \boldsymbol{\xi}^p_{en}) \cdot \dot{\mathbf{g}}^p \ge 0. \tag{37}$$

Based on this inequality we refer to the dissipative microscopic stresses:

$$\tau^p_{dis} = \tau^p - \tau^p_{en}, \qquad \boldsymbol{\xi}^p_{dis} = \boldsymbol{\xi}^p - \boldsymbol{\xi}^p_{en}. \tag{38}$$

This yields the plastic-flow inequality:

$$\delta = \tau^p_{dis}\dot{e}^p + \boldsymbol{\xi}^p_{dis} \cdot \dot{\mathbf{g}}^p \ge 0. \tag{39}$$

3.8 Flow Rule

Within the present framework the flow rule is the microscopic force balance augmented by constitutive relations for the microscopic stresses τ^p and $\boldsymbol{\xi}^p$. We can write this balance in the form:

$$\tau = \tau^p_{en} + \tau^p_{dis} - \mathrm{Div}(\boldsymbol{\xi}^p_{en} + \boldsymbol{\xi}^p_{dis}). \tag{40}$$

The Aifantis theory is based on the following specific assumptions:

1. The defect energy has the quadratic form $\Psi^p = \frac{1}{2}\beta|\mathbf{g}^p|^2$ with $\beta > 0$ constant, so that $\boldsymbol{\xi}^p_{en} = \beta\mathbf{g}^p$ and $\tau^p_{en} \equiv 0$.
2. The constitutive relations for the dissipative microscopic stresses have the purely scalar form $\tau^p_{dis} = Y(e^p)$ and $\boldsymbol{\xi}^p_{dis} \equiv \mathbf{0}$, with coarse-grain flow resistance $Y(e^p) > 0$.

Since $\mathrm{Div}\,\boldsymbol{\xi}^p_{en} = \beta\triangle e^p$, these constitutive relations reduce the general flow rule directly to the Aifantis flow rule:

$$\tau = Y(e^p) - \beta\triangle e^p. \tag{41}$$

The coarse-grain flow-resistance $Y(e^p)$ is therefore dissipative; the nonlocal term $\beta\triangle e^p$ is energetic.

3.9 Microscopically Simple Boundary Conditions

Unlike conventional plasticity theories, the Aifantis flow rule is a partial-differential equation and hence requires concomitant boundary conditions. We consider the boundary ∂B, with outward unit normal $\mathbf{n}$.

We limit our discussion to microscopically simple boundary conditions that result in a null expenditure of microscopic power on the boundary:

$$\dot{e}^p = 0 \quad \text{on } \mathcal{S}_{hard}, \qquad \boldsymbol{\xi}^p \cdot \mathbf{n} = 0 \quad \text{on } \mathcal{S}_{free}, \tag{42}$$

where $\mathcal{S}_{hard}$ and $\mathcal{S}_{free}$ are complementary subsurfaces of ∂B. The microscopically hard condition corresponds to a boundary surface that cannot pass dislocations; the microscopically free condition corresponds to a boundary across which dislocations can flow freely from the body.

3.10 Plastic Free-Energy Balance

Assuming that the microscopically simple boundary conditions are satisfied, we have $(\boldsymbol{\xi}^p \cdot \mathbf{n})\dot{e}^p = 0$ on the boundary ∂B.

Evaluating the temporal change in defect energy and integrating over the body B:

$$\int_B \dot{\overline{\Psi^p(e^p, \mathbf{g}^p)}}\,dv = \int_B (\tau^p_{en}\dot{e}^p + \boldsymbol{\xi}^p_{en} \cdot \nabla\dot{e}^p)\,dv. \tag{43}$$

By utilizing the microscopic force balance $\tau - \tau^p + \mathrm{Div}\,\boldsymbol{\xi}^p = 0$ to expand the integrand, and applying the divergence theorem alongside the null boundary condition, the right side can be rearranged to yield the plastic free-energy balance:

$$\int_B \dot{\overline{\Psi^p(e^p, \mathbf{g}^p)}}\,dv = \int_B \tau\dot{e}^p dv - \int_B (\tau^p_{dis}\dot{e}^p + \boldsymbol{\xi}^p_{dis} \cdot \nabla\dot{e}^p)\,dv. \tag{44}$$

This balance illustrates that the temporal increase in defect energy equals the plastic working minus the non-negative dissipation. Consequently, the temporal increase in defect energy can never exceed the overall plastic working.

4 The Gradient Theory of Gurtin and Anand

This gradient theory differs in many respects from that of Aifantis, chiefly because the codirectionality constraint is not employed. For that reason, the basic kinematical field is the tensorial plastic strain-rate $\dot{\mathbf{E}}^p$ rather than the scalar rate $\dot{e}^p$. Furthermore, this framework represents a rate-dependent gradient theory based on an extension of the conventional virtual-power formulation to include power expended in concert with the plastic strain-rate gradient $\nabla\dot{\mathbf{E}}^p$.

4.1 Kinematic Descriptors

Assuming the plastic spin vanishes, we base our discussion on the "rate-like" descriptors $\dot{\mathbf{u}}$, $\dot{\mathbf{E}}^e$, and $\dot{\mathbf{E}}^p$. These descriptors are restricted by the kinematical constraint:

$$\operatorname{sym}\nabla\dot{\mathbf{u}} = \dot{\mathbf{E}}^e + \dot{\mathbf{E}}^p, \qquad \operatorname{tr}\dot{\mathbf{E}}^p = 0. \tag{45}$$

4.2 Internal Power

Because our goal is a theory that accounts explicitly for plastic-strain gradients, we allow for power expended internally by a third-order microscopic hyperstress $\mathbb{K}^p$ power-conjugate to $\nabla\dot{\mathbf{E}}^p$. We write the internal power in the form:

$$\mathcal{I}(P) = \int_P \Big(\mathbf{T}{:}\dot{\mathbf{E}}^e + \mathbf{T}^p{:}\dot{\mathbf{E}}^p + \mathbb{K}^p \,\vdots\, \nabla\dot{\mathbf{E}}^p\Big)\, dv. \tag{46}$$

Since $\dot{\mathbf{E}}^p$ is symmetric and deviatoric, we assume without loss in generality that $\mathbb{K}^p$ is symmetric and deviatoric in its first two subscripts.

4.3 External Power

The internal power must be balanced by power expended externally by tractions on ∂P and body forces acting within P. Guided by an integral identity based on the divergence theorem, we supplement the conventional external power-expenditure with a higher-order power expenditure involving a microscopic hypertraction $\mathbf{K}(\mathbf{n})$ associated with the hyperstress $\mathbb{K}^p$. We therefore assume that the external power has the form:

$$\mathcal{W}(P) = \int_{\partial P} \mathbf{t}(\mathbf{n})\cdot\dot{\mathbf{u}}\, da + \int_P \mathbf{b}\cdot\dot{\mathbf{u}}\, dv + \int_{\partial P} \mathbf{K}(\mathbf{n}){:}\dot{\mathbf{E}}^p\, da, \tag{47}$$

with $\mathbf{K}(\mathbf{n})$ defined over the body for all time and assumed to be symmetric and deviatoric.

4.4 Power Balance and the Principle of Virtual Power

The principle of virtual power is based on the power balance equating external and internal power:

$$\underbrace{\int_{\partial P}\Big(\mathbf{t}(\mathbf{n})\cdot\dot{\mathbf{u}} + \mathbf{K}(\mathbf{n}){:}\dot{\mathbf{E}}^p\Big)\, da + \int_P \mathbf{b}\cdot\dot{\mathbf{u}}\, dv}_{\mathcal{W}(P)} = \underbrace{\int_P\Big(\mathbf{T}{:}\dot{\mathbf{E}}^e + \mathbf{T}^p{:}\dot{\mathbf{E}}^p + \mathbb{K}^p \,\vdots\, \nabla\dot{\mathbf{E}}^p\Big)\, dv}_{\mathcal{I}(P)}. \tag{48}$$

We consider the fields as virtual fields $\mathcal{V} = (\tilde{\mathbf{u}}, \tilde{\mathbf{E}}^e, \tilde{\mathbf{E}}^p)$ consistent with the kinematical constraint. The principle requires that for any subregion P and any choice of the virtual velocity $\mathcal{V}$:

$$\underbrace{\int_{\partial P}\Big(\mathbf{t}(\mathbf{n})\cdot\tilde{\mathbf{u}} + \mathbf{K}(\mathbf{n}){:}\tilde{\mathbf{E}}^p\Big)\, da + \int_P \mathbf{b}\cdot\tilde{\mathbf{u}}\, dv}_{\mathcal{W}(P,\mathcal{V})} = \underbrace{\int_P\Big(\mathbf{T}{:}\tilde{\mathbf{E}}^e + \mathbf{T}^p{:}\tilde{\mathbf{E}}^p + \mathbb{K}^p \,\vdots\, \nabla\tilde{\mathbf{E}}^p\Big)\, dv}_{\mathcal{I}(P,\mathcal{V})}. \tag{49}$$

4.5 Force Balances

4.5.1 Balance 1: Macroscopic Force Balance

Assuming that $\tilde{\mathbf{u}}$ is arbitrary and that $\tilde{\mathbf{E}}^e = \operatorname{sym}\nabla\tilde{\mathbf{u}}$, so that by the constraint $\tilde{\mathbf{E}}^p \equiv \mathbf{0}$, the virtual power relation yields the classical macroscopic traction condition:

$$\mathbf{t}(\mathbf{n}) = \mathbf{T}\mathbf{n} \tag{50}$$

and the local macroscopic force balance:

$$\operatorname{Div}\mathbf{T} + \mathbf{b} = \mathbf{0}. \tag{51}$$

4.5.2 Balance 2: Microscopic Force Balance

To derive the microscopic force balance, we consider a virtual velocity with $\tilde{\mathbf{E}}^p$ an arbitrary symmetric, deviatoric tensor field, $\tilde{\mathbf{E}}^e = -\tilde{\mathbf{E}}^p$, and $\tilde{\mathbf{u}} \equiv \mathbf{0}$. The power balance reduces to the microscopic virtual-power relation. Applying the divergence theorem identity yields the microscopic traction condition:

$$\mathbf{K}(\mathbf{n}) = \mathbb{K}^p \mathbf{n} \tag{52}$$

and the microscopic force balance:

$$\mathbf{T}_0 = \mathbf{T}^p - \operatorname{Div} \mathbb{K}^p. \tag{53}$$

4.6 Free-Energy Imbalance

Our derivation of the free-energy imbalance leads to the requirement that the temporal increase in free energy of P must be balanced by the power expended on P minus the dissipation. This yields the local free-energy imbalance:

$$\dot{\Psi} - \mathbf{T}{:}\dot{\mathbf{E}}^e - \mathbf{T}^p{:}\dot{\mathbf{E}}^p - \mathbb{K}^p \,\vdots\, \nabla\dot{\mathbf{E}}^p = -\delta \leq 0. \tag{54}$$

4.7 Energetic Constitutive Equations

We assume that the free energy Ψ is the sum of a standard elastic energy and a defect energy $\Psi^p(\mathbf{G})$ dependent on the Burgers tensor $\mathbf{G} = \operatorname{Curl} \mathbf{E}^p$:

$$\Psi = \frac{1}{2}\mathbf{T}{:}\mathbf{E}^e + \Psi^p(\mathbf{G}). \tag{55}$$

By utilizing the chain rule and recognizing that $\nabla\dot{\mathbf{E}}^p$ is symmetric and deviatoric in its first two subscripts, we identify an energetic hyperstress $\mathbb{K}^p_{en}$ associated with temporal changes in the defect energy such that:

$$\dot{\Psi}^p(\mathbf{G}) = \mathbb{K}^p_{en} \,\vdots\, \nabla\dot{\mathbf{E}}^p. \tag{56}$$

4.8 Dissipative Constitutive Equations

Substituting the temporal changes back into the local free-energy imbalance, we define a dissipative hyperstress $\mathbb{K}^p_{dis}$ through the decomposition:

$$\mathbb{K}^p = \mathbb{K}^p_{en} + \mathbb{K}^p_{dis}. \tag{57}$$

This yields the reduced dissipation inequality:

$$\delta = \mathbf{T}^p{:}\dot{\mathbf{E}}^p + \mathbb{K}^p_{dis} \,\vdots\, \nabla\dot{\mathbf{E}}^p \geq 0. \tag{58}$$

We introduce a generalized flow-rate d^p with a constant dissipative length-scale $l > 0$:

$$d^p = \sqrt{|\dot{\mathbf{E}}^p|^2 + l^2|\nabla\dot{\mathbf{E}}^p|^2}. \tag{59}$$

Based on a generalized codirectionality hypothesis requiring that the generalized plastic stress point in the direction of the generalized plastic strain-rate, we formulate the dissipative constitutive equations:

$$\mathbf{T}^p = g(d^p)Y(E^p)\frac{\dot{\mathbf{E}}^p}{d^p}, \qquad \mathbb{K}^p_{dis} = l^2 g(d^p)Y(E^p)\frac{\nabla\dot{\mathbf{E}}^p}{d^p}. \tag{60}$$

4.9 Flow Rule

The microscopic force balance augmented by the constitutive equations forms the flow rule. We place the energetic hyperstress term on the left, since its negative represents a backstress $\mathbf{T}_{back} = -\operatorname{Div} \mathbb{K}^p_{en}$. Substituting the dissipative relations yields the flow rule:

$$\mathbf{T}_0 - \mathbf{T}_{back} = Y(E^p)g(d^p)\frac{\dot{\mathbf{E}}^p}{d^p} - l^2 \operatorname{Div}\left(Y(E^p)g(d^p)\frac{\nabla\dot{\mathbf{E}}^p}{d^p}\right). \tag{61}$$

Given the deviatoric stress $\mathbf{T}_0$, this represents a second-order partial-differential equation for the plastic strain $\mathbf{E}^p$.

4.10 Microscopically Simple Boundary Conditions

Because the flow rule is nonlocal, it must be augmented by appropriate boundary conditions. We limit our discussion to boundary conditions that result in a null expenditure of microscopic power on the boundary. We consider microscopically simple boundary conditions asserting that:

$$\mathbf{E}^p = \mathbf{0} \quad \text{on } \mathcal{S}_{hard} \quad \text{and} \quad \mathbb{K}^p\mathbf{n} = \mathbf{0} \quad \text{on } \mathcal{S}_{free}, \tag{62}$$

where $\mathcal{S}_{hard}$ and $\mathcal{S}_{free}$ are complementary subsurfaces of the boundary ∂B.

4.11 Plastic Free-Energy Balance

Assuming that the microscopically simple boundary conditions are satisfied, we find that the global integration of the defect energy reduces to a balance between plastic working and dissipation:

$$\int_B \dot{\Psi}^p(\mathbf{G})dv = \int_B \mathbf{T}_0{:}\dot{\mathbf{E}}^p dv - \int_B \left(\mathbf{T}^p{:}\dot{\mathbf{E}}^p + \mathbb{K}^p_{dis} \,\vdots\, \nabla\dot{\mathbf{E}}^p\right) dv. \tag{63}$$

Thus, since the dissipation is nonnegative, the temporal increase in defect energy can never exceed the plastic working.

5 Cosserat Theory via the Principle of Virtual Power

5.1 Kinematic Descriptors

In the gradient theory of Gurtin and Anand the basic descriptors are restricted to symmetric rates. In Cosserat theory we introduce the independent microrotation rate $\dot{\boldsymbol{\phi}}$, which is a second-order skew-symmetric tensor.

Because the microstructure rotates independently of the bulk material, the classical symmetric strain rate is replaced by the **asymmetric relative deformation rate $\dot{\mathbf{A}}$**, defined as the mismatch between the macroscopic velocity gradient and the micro-spin:

$$\dot{\mathbf{A}} = \nabla\dot{\mathbf{u}} - \dot{\boldsymbol{\phi}} \tag{64}$$

We assume this relative deformation rate decomposes additively into elastic and plastic parts. The full list of descriptors is therefore:

$$\dot{\mathbf{u}}, \quad \dot{\mathbf{A}}^e, \quad \dot{\mathbf{A}}^p, \quad \dot{\boldsymbol{\phi}},$$

subject to the kinematical constraint:

$$\nabla\dot{\mathbf{u}} - \dot{\boldsymbol{\phi}} = \dot{\mathbf{A}}^e + \dot{\mathbf{A}}^p, \qquad \dot{\boldsymbol{\phi}} \text{ independent and skew-symmetric.} \tag{65}$$

5.2 Internal Power

To capture the Cosserat mechanics, the internal power must account for work done on the asymmetric relative deformation. We define a general **asymmetric Cauchy stress T** and an asymmetric plastic stress $\mathbf{T}^p$. We also allow for power expended internally by a **third-order couple stress** $\mathbb{M}$ power-conjugate to the micro-curvature rate $\nabla\dot{\boldsymbol{\phi}}$.

The **internal power** is therefore written as:

$$\mathcal{I}(P) = \int_P \left(\mathbf{T}{:}\dot{\mathbf{A}}^e + \mathbf{T}^p{:}\dot{\mathbf{A}}^p + \mathbb{M} \,\vdots\, \nabla\dot{\boldsymbol{\phi}}\right) dv. \tag{66}$$

Since $\nabla\dot{\boldsymbol{\phi}}$ is a third-order tensor (the gradient of a second-order tensor), $\mathbb{M}$ must be a third-order tensor. We assume without loss of generality that $\mathbb{M}$ is symmetric and deviatoric in its first two subscripts.

5.3 External Power

The conventional external power is not sufficiently general to accommodate $\mathbb{M} \,\vdots\, \nabla\dot{\boldsymbol{\phi}}$. Guided by the divergence theorem identity:

$$\int_P \mathbb{M} \,\vdots\, \nabla\dot{\boldsymbol{\phi}}\,dv = -\int_P (\mathrm{Div}\,\mathbb{M}) : \dot{\boldsymbol{\phi}}\,dv + \int_{\partial P} (\mathbb{M}\mathbf{n}) : \dot{\boldsymbol{\phi}}\,da, \tag{67}$$

we assume that power is expended externally by a **couple traction** $\mathbf{C}(\mathbf{n})$ — a skew-symmetric second-order tensor conjugate to $\dot{\boldsymbol{\phi}}$ — and a **body couple** $\mathbf{L}$ conjugate to $\dot{\boldsymbol{\phi}}$ within P. The **external power** is:

$$\mathcal{W}(P) = \int_{\partial P} \mathbf{t}(\mathbf{n}) \cdot \dot{\mathbf{u}}\,da + \int_P \mathbf{b} \cdot \dot{\mathbf{u}}\,dv + \int_{\partial P} \mathbf{C}(\mathbf{n}) : \dot{\boldsymbol{\phi}}\,da + \int_P \mathbf{L} : \dot{\boldsymbol{\phi}}\,dv. \tag{68}$$

5.4 Power Balance and the Principle of Virtual Power

The principle of virtual power equates external and internal power: $\mathcal{W}(P) = \mathcal{I}(P)$. We consider the fields as virtual fields $\tilde{\mathbf{u}}$, $\tilde{\mathbf{A}}^e$, $\tilde{\mathbf{A}}^p$, $\tilde{\boldsymbol{\phi}}$. A **generalized virtual velocity** is the list $\mathcal{V} = \big(\tilde{\mathbf{u}}, \tilde{\mathbf{A}}^e, \tilde{\mathbf{A}}^p, \tilde{\boldsymbol{\phi}}\big)$ consistent with the kinematical constraint:

$$\nabla\tilde{\mathbf{u}} - \tilde{\boldsymbol{\phi}} = \tilde{\mathbf{A}}^e + \tilde{\mathbf{A}}^p. \tag{69}$$

The **principle of virtual power** requires that for any subregion P and any virtual velocity $\mathcal{V}$:

$$\underbrace{\int_{\partial P}\Big(\mathbf{t}(\mathbf{n}) \cdot \tilde{\mathbf{u}} + \mathbf{C}(\mathbf{n}) : \tilde{\boldsymbol{\phi}}\Big)\,da + \int_P \Big(\mathbf{b} \cdot \tilde{\mathbf{u}} + \mathbf{L} : \tilde{\boldsymbol{\phi}}\Big)\,dv}_{\mathcal{W}(P,\mathcal{V})} = \underbrace{\int_P \Big(\mathbf{T} : \tilde{\mathbf{A}}^e + \mathbf{T}^p : \tilde{\mathbf{A}}^p + \mathbb{M} \,\vdots\, \nabla\tilde{\boldsymbol{\phi}}\Big)\,dv}_{\mathcal{I}(P,\mathcal{V})}. \tag{70}$$

5.5 Force and Moment Balances

5.5.1 Balance 1: Macroscopic Force Balance

We choose the virtual velocity:

$$\tilde{\mathbf{A}}^e = \nabla\tilde{\mathbf{u}}, \qquad \tilde{\mathbf{A}}^p \equiv \mathbf{0}, \qquad \tilde{\boldsymbol{\phi}} \equiv \mathbf{0}, \tag{71}$$

with $\tilde{\mathbf{u}}$ arbitrary. All couple terms vanish, and the balance becomes:

$$\int_{\partial P} \mathbf{t}(\mathbf{n}) \cdot \tilde{\mathbf{u}}\,da + \int_P \mathbf{b} \cdot \tilde{\mathbf{u}}\,dv = \int_P \mathbf{T} : \nabla\tilde{\mathbf{u}}\,dv. \tag{72}$$

Applying the divergence theorem and the fundamental lemma yields the **traction condition** and the **macroscopic force balance**:

$$\mathbf{t}(\mathbf{n}) = \mathbf{T}\mathbf{n}, \qquad \mathrm{Div}\,\mathbf{T} + \mathbf{b} = \mathbf{0}. \tag{73}$$

Note that $\mathbf{T}$ *is now the full asymmetric Cauchy stress tensor.*

5.5.2 Balance 2: Plastic Microscopic Balance

We choose the virtual velocity:

$$\tilde{\mathbf{u}} \equiv \mathbf{0}, \qquad \tilde{\mathbf{A}}^e = -\tilde{\mathbf{A}}^p, \qquad \tilde{\boldsymbol{\phi}} \equiv \mathbf{0}, \tag{74}$$

with $\tilde{\mathbf{A}}^p$ an arbitrary asymmetric tensor field. All couple terms vanish, and the balance reduces to:

$$\int_P (\mathbf{T}^p - \mathbf{T}) : \tilde{\mathbf{A}}^p\,dv = 0. \tag{75}$$

Since this holds for all $\tilde{\mathbf{A}}^p$, the fundamental lemma yields the **plastic microscopic balance**:

$$\mathbf{T} = \mathbf{T}^p. \tag{76}$$

5.5.3 Balance 3: Microscopic Moment Balance

We choose the virtual velocity:

$$\tilde{\mathbf{u}} \equiv \mathbf{0}, \qquad \tilde{\mathbf{A}}^p \equiv \mathbf{0}, \qquad \tilde{\boldsymbol{\phi}} \text{ arbitrary (skew-symmetric).} \tag{77}$$

By the constraint $\nabla\tilde{\mathbf{u}} - \tilde{\boldsymbol{\phi}} = \tilde{\mathbf{A}}^e + \tilde{\mathbf{A}}^p$, this forces $\tilde{\mathbf{A}}^e = -\tilde{\boldsymbol{\phi}}$. The macroscopic external forces vanish, and the balance reduces to:

$$\int_{\partial P} \mathbf{C}(\mathbf{n}){:}\tilde{\boldsymbol{\phi}}\, da + \int_P \mathbf{L}{:}\tilde{\boldsymbol{\phi}}\, dv = \int_P \left(-\mathbf{T}{:}\tilde{\boldsymbol{\phi}} + \mathbb{M} \vdots \nabla\tilde{\boldsymbol{\phi}}\right) dv. \tag{78}$$

Because $\tilde{\boldsymbol{\phi}}$ is skew-symmetric, the contraction $\mathbf{T}{:}\tilde{\boldsymbol{\phi}}$ isolates the skew-symmetric part of the stress: $\operatorname{skew}\mathbf{T}{:}\tilde{\boldsymbol{\phi}}$. Applying the divergence theorem identity to the right-hand side and rearranging yields:

$$\int_{\partial P} (\mathbf{C}(\mathbf{n}) - \mathbb{M}\mathbf{n}) : \tilde{\boldsymbol{\phi}}\, da + \int_P (\operatorname{Div}\mathbb{M} + \operatorname{skew}\mathbf{T} + \mathbf{L}) : \tilde{\boldsymbol{\phi}}\, dv = 0. \tag{79}$$

Since this holds for all $\tilde{\boldsymbol{\phi}}$ and all P, the fundamental lemma yields the **couple traction condition** $\mathbf{C}(\mathbf{n}) = \mathbb{M}\mathbf{n}$ and the **microscopic moment balance**:

$$\operatorname{Div}\mathbb{M} + \operatorname{skew}\mathbf{T} + \mathbf{L} = \mathbf{0}. \tag{80}$$

This perfectly recovers the hallmark Cosserat coupling: internal microstructural friction ($\operatorname{Div}\mathbb{M}$*) directly induces an asymmetry in the macroscopic Cauchy stress (*$\operatorname{skew}\mathbf{T}$*).*

5.6 Free-Energy Imbalance & Constitutive Equations

We treat the microrotation gradient $\nabla\dot{\boldsymbol{\phi}}$ as the micro-curvature rate tensor $\dot{\mathbb{K}} = \dot{\mathbb{K}}^e + \dot{\mathbb{K}}^p$. The local free-energy imbalance requires $\dot{\Psi} - \mathcal{I}_{density} = -\delta \leq 0$:

$$\dot{\Psi} - \mathbf{T}{:}\dot{\mathbf{A}}^e - \mathbb{M} \vdots \dot{\mathbb{K}}^e - \mathbf{T}^p{:}\dot{\mathbf{A}}^p - \mathbb{M} \vdots \dot{\mathbb{K}}^p = -\delta \leq 0 \tag{81}$$

Assuming $\Psi = \hat{\Psi}(\mathbf{A}^e, \mathbb{K}^e)$, the **energetic state equations** are extracted via the chain rule:

$$\mathbf{T} = \frac{\partial\hat{\Psi}}{\partial\mathbf{A}^e}, \qquad \mathbb{M}_{en} = \frac{\partial\hat{\Psi}}{\partial\mathbb{K}^e}. \tag{82}$$

This leaves the **reduced dissipation inequality** governing plastic flow:

$$\delta = \mathbf{T}^p{:}\dot{\mathbf{A}}^p + \mathbb{M}_{dis} \vdots \dot{\mathbb{K}}^p \geq 0. \tag{83}$$

Given the effective generalized plastic flow rate $d^p = \sqrt{|\dot{\mathbf{A}}^p|^2 + l^2|\dot{\mathbb{K}}^p|^2}$, the dissipative stresses are defined as:

$$\mathbf{T}^p = Y(A^p)g(d^p)\frac{\dot{\mathbf{A}}^p}{d^p}, \qquad \mathbb{M}_{dis} = l^2 Y(A^p)g(d^p)\frac{\dot{\mathbb{K}}^p}{d^p}. \tag{84}$$

5.7 Flow Rules and Free-Energy Balance

Substituting the constitutive equations into the balances yields the translational flow rule ($\mathbf{T} = \mathbf{T}^p$) and the rotational flow rule:

$$-\operatorname{Div}\mathbb{M}_{en} - \operatorname{skew}\mathbf{T} - \mathbf{L} = l^2 \operatorname{Div}\left(Y(A^p)g(d^p)\frac{\dot{\mathbb{K}}^p}{d^p}\right). \tag{85}$$

Assuming microscopically simple boundary conditions ($\dot{\boldsymbol{\phi}} = \mathbf{0}$ or $\mathbb{M}\mathbf{n} = \mathbf{0}$), global integration yields the plastic free-energy balance:

$$\int_B \dot{\Psi}^p dv = \int_B \mathbf{T}{:}\dot{\mathbf{A}}^p dv - \int_B \left(\mathbf{T}^p{:}\dot{\mathbf{A}}^p + \mathbb{M}^p_{dis} \vdots \nabla\dot{\phi}^p\right) dv. \tag{86}$$

The temporal increase in defect energy can never exceed the plastic working.

6 Micromorphic Theory via the Principle of Virtual Power

Micromorphic theory represents the most radical departure from classical mechanics by breaking the assumption of microstructural rigidity. In this framework, material points are treated as deformable micro-volumes that can independently stretch, shear, and rotate.

6.1 Kinematic Descriptors

The kinematics are described by the macroscopic translational velocity $\dot{\mathbf{u}}$ and an independent **micro-velocity gradient** $\dot{\mathbf{P}}$. Unlike Cosserat theory, which restricts the microstructure to rigid rotations, $\dot{\mathbf{P}}$ is a full, asymmetric second-order tensor encompassing micro-stretch, micro-shear, and micro-rotation.

The full list of descriptors is:

$$\dot{\mathbf{u}}, \quad \dot{\mathbf{E}}^e, \quad \dot{\mathbf{E}}^p, \quad \dot{\mathbf{P}}$$

To bridge the macroscopic and microscopic kinematics, we define the **relative deformation rate**, which measures how much the macroscopic continuum deformation deviates from the internal micro-deformation:

$$\dot{\mathbf{A}} = \nabla\dot{\mathbf{u}} - \dot{\mathbf{P}} \tag{87}$$

We assume the macroscopic velocity gradient decomposes into elastic and plastic parts as standard: $\nabla\dot{\mathbf{u}} = \dot{\mathbf{E}}^e + \dot{\mathbf{E}}^p$.

6.2 Internal Power

To accommodate these complex internal kinematics, we assume the macroscopic force-stress $\mathbf{T}$ does work on the relative deformation rate. Additionally, we introduce a **micro-stress tensor S** (second-order) power-conjugate to the micro-deformation $\dot{\mathbf{P}}$, and a **micro-hyperstress** $\mathbb{M}$ (third-order) power-conjugate to the micro-curvature rate $\nabla\dot{\mathbf{P}}$.

The **internal power** is written as:

$$\mathcal{I}(P) = \int_P \Big(\mathbf{T}{:}(\nabla\dot{\mathbf{u}} - \dot{\mathbf{P}}) + \mathbf{S} : \dot{\mathbf{P}} + \mathbb{M} \,\vdots\, \nabla\dot{\mathbf{P}}\Big)\, dv. \tag{88}$$

Substituting our elastic-plastic split ($\nabla\dot{\mathbf{u}} = \dot{\mathbf{E}}^e + \dot{\mathbf{E}}^p$), the power takes the explicit form:

$$\mathcal{I}(P) = \int_P \Big(\mathbf{T}{:}\dot{\mathbf{E}}^e + \mathbf{T}{:}\dot{\mathbf{E}}^p + (\mathbf{S} - \mathbf{T}) : \dot{\mathbf{P}} + \mathbb{M} \,\vdots\, \nabla\dot{\mathbf{P}}\Big)\, dv. \tag{89}$$

6.3 External Power

The external power must account for forces conjugate to the deformable microstructure. We introduce a **micro-traction C(n)** (second-order) and a **micro-body force L** (second-order) conjugate to $\dot{\mathbf{P}}$:

$$\mathcal{W}(P) = \int_{\partial P} \Big(\mathbf{t}(\mathbf{n}) \cdot \dot{\mathbf{u}} + \mathbf{C}(\mathbf{n}) : \dot{\mathbf{P}}\Big)\, da + \int_P \Big(\mathbf{b} \cdot \dot{\mathbf{u}} + \mathbf{L} : \dot{\mathbf{P}}\Big)\, dv. \tag{90}$$

6.4 Force and Micro-Balances

Applying the principle of virtual power ($\mathcal{W} = \mathcal{I}$) for arbitrary virtual fields yields the governing balance equations:

Balance 1: Macroscopic Balance ($\tilde{\mathbf{u}}$ arbitrary, $\tilde{\mathbf{P}} \equiv \mathbf{0}$):

$$\operatorname{Div}\mathbf{T} + \mathbf{b} = \mathbf{0}, \qquad \mathbf{t}(\mathbf{n}) = \mathbf{T}\mathbf{n} \tag{91}$$

Balance 2: Micro-Deformation Balance ($\tilde{\mathbf{P}}$ arbitrary, $\tilde{\mathbf{u}} \equiv \mathbf{0}$): Applying the divergence theorem to $\mathbb{M} \,\vdots\, \nabla\tilde{\mathbf{P}}$ yields the higher-order balance governing the internal deformable microstructure:

$$\operatorname{Div}\mathbb{M} + \mathbf{T} - \mathbf{S} + \mathbf{L} = \mathbf{0}, \qquad \mathbf{C}(\mathbf{n}) = \mathbb{M}\mathbf{n} \tag{92}$$

Notice how the macroscopic stress $\mathbf{T}$ acts as an internal driving force on the micro-deformation due to the relative strain coupling.

6.5 Free-Energy Imbalance

We assume an additive decomposition of the third-order micro-curvature rate into elastic and plastic parts: $\nabla\dot{\mathbf{P}} = \dot{\mathbb{K}}^e + \dot{\mathbb{K}}^p$. The local free-energy imbalance requires that $\dot{\Psi} - \mathcal{I}_{density} = -\delta \leq 0$:

$$\dot{\Psi} - \mathbf{T}{:}\dot{\mathbf{E}}^e - \mathbb{M} \,\vdots\, \dot{\mathbb{K}}^e - \mathbf{T}{:}\dot{\mathbf{E}}^p - (\mathbf{S} - \mathbf{T}){:}\dot{\mathbf{P}} - \mathbb{M} \,\vdots\, \dot{\mathbb{K}}^p = -\delta \leq 0 \tag{93}$$

6.6 Energetic and Dissipative Constitutive Equations

Assuming the free energy depends on the recoverable elastic strains and the micro-curvature, $\Psi = \hat{\Psi}(\mathbf{E}^e, \mathbb{K}^e, \mathbf{P})$, we extract the **energetic state equations**:

$$\mathbf{T} = \frac{\partial\hat{\Psi}}{\partial\mathbf{E}^e}, \qquad \mathbb{M}_{en} = \frac{\partial\hat{\Psi}}{\partial\mathbb{K}^e}, \qquad \mathbf{S}_{en} - \mathbf{T} = \frac{\partial\hat{\Psi}}{\partial\mathbf{P}} \tag{94}$$

The reduced dissipation inequality governing micromorphic plastic flow isolates the dissipative parts of the stresses:

$$\delta = \mathbf{T}{:}\dot{\mathbf{E}}^p + \mathbf{S}_{dis}{:}\dot{\mathbf{P}} + \mathbb{M}_{dis} \,\vdots\, \dot{\mathbb{K}}^p \geq 0 \tag{95}$$

6.7 Flow Rules

Defining a generalized flow rate d^p for micromorphic media and substituting dissipative constitutive relations back into the micro-balance, we obtain the **Micromorphic Flow Rule**:

$$-\operatorname{Div}\mathbb{M}_{en} + \mathbf{S}_{en} - \mathbf{T} - \mathbf{L} = l^2 \operatorname{Div}\left(Y(E^p) g(d^p) \frac{\dot{\mathbb{K}}^p}{d^p} \right) \tag{96}$$

6.8 Plastic Free-Energy Balance

Assuming the satisfaction of microscopically simple boundary conditions (where either $\dot{\mathbf{P}} = \mathbf{0}$ or $\mathbb{M}\mathbf{n} = \mathbf{0}$ on the boundary), the global integration yields:

$$\int_B \dot{\Psi}^p dv = \int_B \left(\mathbf{T}{:}\dot{\mathbf{E}}^p + (\mathbf{S} - \mathbf{T}){:}\dot{\mathbf{P}} + \mathbb{M}{:}\dot{\mathbb{K}}^p \right) dv - \int_B \delta dv \tag{97}$$

This ensures the stored energy in the deformable microstructure never exceeds the total generalized plastic working performed on the micromorphic system.

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