Final Project: MPC Design

ME C231A : Model Predictive Control
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1 Abstract

Presentation Video: https://youtu.be/sfoAw0bwIM0

A common challenge in both human driving and automated vehicle control is turning a car around in a narrow roadway. On wider roads, a simple three-point turn is typically sufficient to complete the maneuver. However, as the available road width decreases, more complex driving strategies are required to successfully rotate the vehicle. The goal of this report is to develop a generalized Model Predictive Control (MPC) framework for optimizing the vehicle rotation process under such constrained conditions. The proposed MPC formulation incorporates both hard and soft constraints and serves as a controller for the vehicle's acceleration and steering angle. For the purpose of the system, the car we will control is a 2025 Honda Accord with a steering limit of 35° and a wheelbase of 2.82 meters [2].

In our simulation, the vehicle is initialized at the center of the road, facing forward with a heading angle of 0°. The controller then computes an optimal sequence of steering angles and accelerations to reorient the vehicle by 180°, returning it to the centerline while facing the opposite direction. To evaluate the performance of the proposed controller, multiple simulations are conducted across a range of road widths. In wider roads, there is a sufficient amount space for the vehicle to execute smoother turning motions, avoiding the need to constantly switch from going forward and backwards. On the other hand, for thinner roads, it is often infeasible for the car to successfully do a U-turn without going over the bounds of the road. Therefore, we will also implement a slack cost that allows the car to go slightly over the edge of the road in order to make it possible to turn around in thinner roads.

2 Theory

2.1 Problem Setup

We bring the setup of this problem by defining the controlled object dynamics with the form

$$x_{k+1} = f(x_k, u_k) \tag{1}$$

where x_k denotes the object's state at time-step k. The problem is initialized at the state x_0 and controlled to its terminal state x_F. The inputs are to be defined as u_k and the dynamics of the problem, to be elaborated on later, will be represented by f. The constraints for the problem are represented by

$$h(x_k, u_k) \leq 0 \tag{2}$$

The goal is to optimize the objective cost function

$$J = \sum_{k=0}^{N} l(x_k, u_k) \tag{3}$$

over a horizon of length N, with l being the stage cost.

2.2 Control Problem

Within the control problem, at some state x_k, $\mathbb{E}(x_k)$ denotes the space that the controlled object occupies, in this case being the body of the car. We expand the constraints to include one for collision avoidance.

$$\mathbb{E}(x_k) \cap \mathbb{O}^{(m)} = \emptyset \; \forall m = 1, ..., M \tag{4}$$

This constraint represents the intersection of the controlled object $\mathbb{E}(x_k)$ and M amount of obstacles, $\mathbb{O}^{(m)}$, which are convex sets with non-empty interior that are represented as

$$\mathbb{O}^{(m)} = y \in \mathbb{R}^n : A^{(m)} y < \kappa b^{(m)} \tag{5}$$

where $A^{(m)}$ and $b^{(m)}$ define the polytopes of each obstacle, m.

Since we are going to be looking at the control of objects, which are full-dimensional, $\mathbb{E}(x_k)$ will be represented by the following function,

$$\mathbb{E}(x_k) = R(x_k)\mathbb{B} + t(x_k), \mathbb{B} := y : Gy \leq \kappa g \tag{6}$$

κ helps define the shape of the initial set $\mathbb{B}$ and $R(*)$ being the rotation matrix to help get the position of the controlled body at time step k.

After these definitions, we now have a constrained finite-horizon optimal control problem with collision avoidance that will be set up as follows

$$\begin{aligned} \min_{\mathbf{x},\, \mathbf{u}} \quad & \sum_{k=0}^{N} \ell(x_k, u_k) \\ \text{s.t.} \quad & x_0 = x_S, \qquad x_{N+1} = x_F, \\ & x_{k+1} = f(x_k, u_k), \\ & h(x_k, u_k) \leq 0, \\ & \mathbb{E}(x_k) \cap \mathbb{O}^{(m)} = \emptyset, \end{aligned} \tag{7}$$

2.3 Collision Avoidance

For the problem of avoiding collisions we have to look at the signed distance formula shown in (8) to have a way to quantify the intersection of the controlled object $\mathbb{E}(x_k)$ and some obstacle $\mathbb{O}^{(m)}$.

$$sd(\mathbb{E}(x_k), \mathbb{O}) := dist(\mathbb{E}(x_k), \mathbb{O}) - pen(\mathbb{E}(x_k), \mathbb{O}) \tag{8}$$

with both dist(.,.) and pen(.,.) being defined by:

$$dist(\mathbb{E}(x_k), \mathbb{O}) := \inf_t ||t|| : \mathbb{E}(x) + t \in \mathbb{O}, \quad (9)$$

$$pen(\mathbb{E}(x_k), \mathbb{O}) := \inf_t ||t|| : \mathbb{E}(x) + t \notin \mathbb{O}, \quad (10)$$

When a collision occurs the signed distance will be zero if their intersection has empty interior (just touching), and negative if the intersection is non-negative interior. To allow for retaining ease of solving within the optimization problem, $sd(\mathbb{E}(x_k), \mathbb{O}) > 0$ must not be enforced in the current state and will need to go through reformulation as outlined in [1] and will result in

$$\begin{aligned}
\min_{\mathbf{x}, \mathbf{u}, s, \lambda, \mu} \quad & \left[\sum_{k=0}^{N} \ell(x_k, u_k) + \kappa \cdot \sum_{m=0}^{M} s_k^{(m)} \right] \\
\text{s.t.} \quad & x_0 = x_S, \qquad x_{N+1} = x_F, \\
& x_{k+1} = f(x_k, u_k), h(x_k, u_k) \leq 0, \\
& -g' \mu_k^{(m)} + (A^{(m)} t(x_k) - b^{(m)})' \lambda_k^{(m)} > -s_k^{(m)}, \\
& G' \mu_k^{(m)} + R(x_k)' A^{(m)'} \lambda_k^{(m)} = 0 \\
& ||A^{(m)'} \lambda_k^{(m)}||_* = 1, s_k^{(m)} \geq 0, \\
& \lambda_k^{(m)} \succeq \kappa^* 0, \mu_k^{(m)} \succeq \bar{\kappa}^* 0
\end{aligned} \quad (11)$$

Within this definition, there is the addition of a new decision variable, $s_k^{(m)}$, that is the slack variable at time k associated with obstacle $\mathbb{O}^{(m)}$. This variable relaxes the signed-distance-based separated between the controlled object and obstacles. κ is a weight variable that incentivizes keeping the slack variable small and only activate when absolutely necessary. To accomplish this reformulation, the collision-avoidance constraint is converted into a dual variable problem with $\lambda_k^{(m)}$ and $\mu_k^{(m)}$ being the dual variables that are associated with an obstacle $\mathbb{O}^{(m)}$ at some time step k. Also added in is the presence of a new term in the cost function. The $\kappa \cdot \sum_{m=0}^{M} s_k^{(m)}$ term is how the problem discourages large slack variable choices and only have them added when absolutely necessary. The larger the κ, the more it will cost to soften the constraints.

3 Model

3.1 System Dynamics

The system dynamics described in 1 are representative of the kinematic bicycle model which can be applied to this car. This is a viable assumption for the situation we are controlling for because of the low speeds and no dynamics. It can be expanded into

$$\begin{aligned}
\dot{X} &= v \cos(\psi) \\
\dot{Y} &= v \sin(\psi) \\
\dot{\psi} &= v \frac{\tan \delta}{L} \\
\dot{v} &= a
\end{aligned} \quad (12)$$

X and Y denote the coordinates of the center of mass. ψ is the car's heading angle, or the angle at which the velocity of the car's center of mass, v, is relative to the positive x-axis. The parameters of the car include the wheelbase length, $L = 2.82m$. We impose actuator limits to be $|a| < 1\frac{m}{s^2}$ and $|\delta| < 0.61rad$, and limit the car's velocity to $|v| < 1.5\frac{m}{s}$. For viability in MPC, these dynamics and discretized using the forward Euler method.

3.2 Soft Constraints

While in many cases, there is ample room to maneuver the car and the control problem will be feasible, we need to address the times when it becomes infeasible. The formulation seen in (7) will work for these collision free cases, it needs to be reformulated to allow for a way to minimize collisions when it would otherwise be infeasible. The formulation (11) accounts for this through the use of the slack variables that act to quantify and soften the constraint against overlapping the controlled object and obstacle. Prior to the implementation of the soft constraints, the problem formulation in 7 would only remain feasible in a collision-free scenario. This gives it the ability to add slack variables to minimize the collisions when absolutely necessary.

4 MPC Design and Implementation

4.1 Hard-Constraint MPC

We first implemented:

$$\begin{aligned}
\min_{\{x_k, u_k\}} \quad & \sum_{k=0}^{N-1} \ell(x_k, u_k) + V_f(x_N) \\
\text{s.t.} \quad & x_{k+1} = f(x_k, u_k), \\
& h(x_k, u_k) \leq 0, \\
& x_0 = x_{\text{init}}.
\end{aligned} \quad (13)$$

Here, lane boundaries, velocity limits, and the car footprint must be satisfied exactly. **Result:** For a three-point turn in a 3 meter lane, this problem becomes infeasible.

4.2 Soft-Constraint MPC

To expand on our problem specific use of slack variables, we introduce constraint specific variables

$$s_k = \begin{bmatrix} s_{k,1} \\ s_{k,2} \\ s_{k,3} \\ s_{k,4} \end{bmatrix} \geq 0,$$

that are correlated to:

- lower boundary
- upper boundary
- minimum velocity
- maximum velocity.

To get the smooth responses of our MPC, we relaxed constraints as

$$h_i(x_k) \le s_{k,i},$$

and penalized slacks quadratically:

$$\ell(x_k, u_k, s_k) = (x_k - x_r)^\top Q(x_k - x_r) + u_k^\top R u_k + \kappa \|s_k\|^2.$$

4.3 Distance-Maximizing Terminal Cost

To make the controller "push forward" during the maneuver, we define:

$$V_f(x_N) = (x_N - x_r)^\top P(x_N - x_r) - \alpha X_N,$$

with $\alpha = 50$. This encourages maximizing forward displacement subject to road limits.

4.4 Results

The soft-constraint MPC:

- remains feasible throughout the maneuver
- respects road boundaries except for minimal slack-induced relaxation
- produces realistic acceleration and steering profiles
- achieves large forward displacement due to the $-\alpha X_N$ term.

Meanwhile the hard-constraint MPC failed to find a feasible solution.

A simulation of the MPC controller on a large road (width = 8 meters) can be seen in Figure 1. It is able to perform a 4-point turn around without any interference.

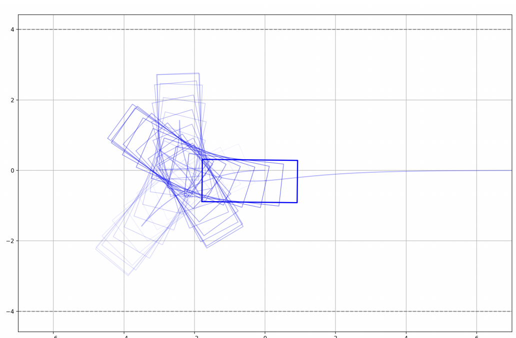

Figure 1: Simulation on a large road

A simulation of the MPC controller on a small road (width = 3 meters) can be seen in Figure 2. This simulation activates the slack variables and turns the hard constraints into soft. The car is still able to turn while minimizing the penetration of with the road boundaries. This allows for the controller to still generate a trajectory that is feasible when otherwise it would be infeasible without penetration minimization.

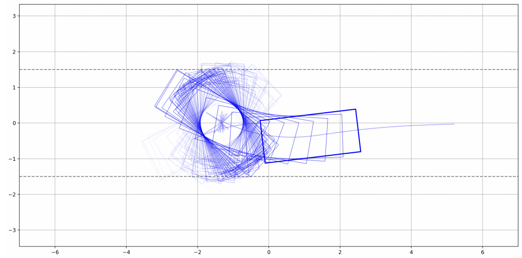

Figure 2: Simulation on a small road

An intermediate road width of 4.5 meters was also tested and shown in Figure 3 which resulted in a 6 point turn.

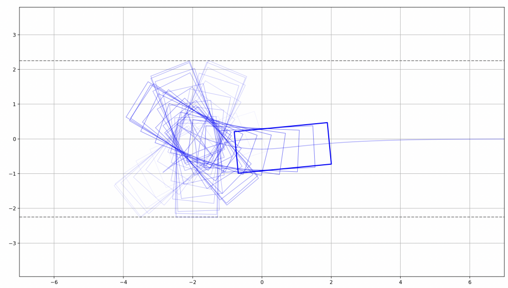

Figure 3: Simulation on a medium road

References

[1] Zhang, X; Liniger, A; Borrelli, F. (2017). "Optimization-Based Collision Avoidance".

[2] "2025 Honda Accord Exterior" `https://www.hondaofcovington.com/2025-honda-accord-exterior/`